Problema săptămânii 126
Dacă $a, b, c, d \in [0, 1]$, aflaţi valoarea maximă a expresiei

$$|(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)|.$$

antrenament loturi, Franţa, 2014

Soluţie:
Fie $E(a, b, c, d) = |(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)|$. Dacă două sau mai multe dintre numere sunt egale, expresia ia valoarea 0 care, în mod evident, nu este cea mai mare posibilă. Expresia fiind simetrică, putem presupune $a > b > c > d$. Atunci este evident că $E(a, b, c, d) \leq E(1, b, c, 0) = (1-b)(1-c)(b-c)bc$.

Din inegalitatea mediilor, $(1-b)c \leq \left(\frac{1-b+c}{2}\right)^2$ şi $(1-c)b \leq \left(\frac{1-c+b}{2}\right)^2$, cu egalitate dacă $b + c = 1$. Notând $t = b - c$, avem că $t \in (0, 1)$ şi

$$E(a, b, c, d) \leq (1-b)c \cdot (1-c)b \cdot (b-c) \leq \frac{(1-t)^2}{4} \cdot \frac{(1+t)^2}{4} \cdot t = \frac{t(1-t^2)^2}{16}.$$

Dar, din inegalitatea mediilor,

$$4t^2(1-t^2)^4 \leq \left(\frac{4t^2 + (1-t^2) + (1-t^2) + (1-t^2) + (1-t^2)}{5}\right)^5 = \left(\frac{4}{5}\right)^5,$$

deci $t(1-t^2)^2 \leq \frac{2^4}{25\sqrt{5}}$, cu egalitate dacă $4t^2 = 1 - t^2$, adică pentru $t = \frac{1}{\sqrt{5}}$. Am obţinut aşadar că $E(a, b, c, d) \leq \frac{\sqrt{5}}{125}$, cu egalitate dacă $b - c = \frac{1}{\sqrt{5}}$, $b + c = 1$, $a = 1$, $d = 0$, deci pentru $a = 1$, $b = \frac{5+\sqrt{5}}{10}$, $c = \frac{5-\sqrt{5}}{10}$, $d = 0$.

Problem of the week no. 126
Find the maximum value of the expression $|(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)|$ when $a, b, c, d \in [0, 1]$.

French training, 2014

Solution:
Put $E(a, b, c, d) = |(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)|$. If two or more variables are equal, the expression E takes the value 0 which, clearly, is not the largest possible. The expression being symmetric, we may assume $a > b > c > d$. It is then clear that $E(a, b, c, d) \leq E(1, b, c, 0) = (1-b)(1-c)(b-c)bc$.

From the AM-GM inequality, $(1-b)c \leq \left(\frac{1-b+c}{2}\right)^2$ and $(1-c)b \leq \left(\frac{1-c+b}{2}\right)^2$, with equality if $b + c = 1$. Denoting $t = b - c$, we have $t \in (0, 1)$ and

$$E(a, b, c, d) \leq (1-b)c \cdot (1-c)b \cdot (b-c) \leq \frac{(1-t)^2}{4} \cdot \frac{(1+t)^2}{4} \cdot t = \frac{t(1-t^2)^2}{16}.$$

But using AM-GM again, gives

$$4t^2(1-t^2)^4 \leq \left(\frac{4t^2 + (1-t^2) + (1-t^2) + (1-t^2) + (1-t^2)}{5}\right)^5 = \left(\frac{4}{5}\right)^5,$$

hence $t(1-t^2)^2 \leq \dfrac{2^4}{25\sqrt{5}}$, with equality if $4t^2 = 1-t^2$, i.e. for $t = \dfrac{1}{\sqrt{5}}$. We have thus proven that $E(a,b,c,d) \leq \dfrac{\sqrt{5}}{125}$, with equality being attained when $b-c = \dfrac{1}{\sqrt{5}}$, $b+c=1$, $a=1$, $d=0$, i.e. for $a=1$, $b = \dfrac{5+\sqrt{5}}{10}$, $c = \dfrac{5-\sqrt{5}}{10}$, $d=0$.