Problema săptămânii 121

Fie $a, b, c \in (0,1)$ astfel încât $abc = (1-a)(1-b)(1-c)$. Demonstraţi că $abc \leq \dfrac{1}{8}$.

Soluţia 1:

Avem $\sqrt[3]{abc} = \sqrt[3]{(1-a)(1-b)(1-c)} \leq \dfrac{1-a+1-b+1-c}{3} = 1 - \dfrac{a+b+c}{3} \leq 1-\sqrt[3]{abc}$, de unde $\sqrt[3]{abc} \leq \dfrac{1}{2}$, adică $abc \leq \dfrac{1}{8}$. Egalitatea are loc dacă $a = b = c = \dfrac{1}{2}$.

Soluţia 2:

Inegalitatea de demonstrat este echivalentă cu $(abc)^2 \leq \dfrac{1}{64}$, deci cu $abc(1-a)(1-b)(1-c) \leq \dfrac{1}{64}$. Ea rezultă din înmulţirea relaţiilor $0 < a(1-a) \leq \dfrac{1}{4}$, $0 < b(1-b) \leq \dfrac{1}{4}$ şi $0 < c(1-c) \leq \dfrac{1}{4}$.

Sau, aşa cum propune *Ervin Macić*, ea rezultă şi din inegalitatea mediilor: avem $\sqrt[6]{abc(1-a)(1-b)(1-c)} \leq \dfrac{a+b+c+1-a+1-b+1-c}{6} = \dfrac{1}{2}$.

Soluţia 3:

Notând $x = \dfrac{a}{1-a}$, $y = \dfrac{b}{1-b}$, $z = \dfrac{c}{1-c}$, avem $x, y, z > 0$ cu $xyz = 1$, iar $a = \dfrac{x}{1+x}$, $b = \dfrac{y}{1+y}$, $c = \dfrac{z}{1+z}$, deci inegalitatea de demonstrat revine la $(1+x)(1+y)(1+z) \geq 8xyz$ dacă $x, y, z > 0$ satisfac $xyz = 1$. Ori $1 + x \geq 2\sqrt{x}$, $1 + y \geq 2\sqrt{y}$ şi $1 + z \geq 2\sqrt{z}$, deci $(1+x)(1+y)(1+z) \geq 8\sqrt{xyz} = 8xyz$.

Remarcă: Prin decondiţionarea standard, $x = \frac{m}{n}$, $y = \frac{n}{p}$, $z = \frac{p}{m}$, inegalitatea $(1+x)(1+y)(1+z) \geq 8xyz$ revine la cunoscuta inegalitate $(m+n)(n+p)(p+m) \geq 8mnp$ (care se demonstrează la fel ca cea de mai sus). Această inegalitate îi este uneori atribuită lui Cesàro. Din ea rezultă imediat inegalitatea lui Euler ($R \geq 2r$).

Soluţia 4: (*Ervin Macić*)

Cum $a, b, cin(0,1)$, avem $x^{1/2} < x^{1/3}$, deci $\sqrt[3]{abc} > \sqrt{abc}$, sau $\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)}$. Putem demonstra $\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1$ cu inegalitatea mediilor şi atunci, cum $\sqrt[3]{abc} = \sqrt[3]{(1-a)(1-b)(1-c)}$, obţinem că $2\sqrt[3]{abc} \leq 1$ şi concluzia $abc \leq \frac{1}{8}$.

Problem of the week no. 121

Consider $a, b, c \in (0,1)$ such that $abc = (1-a)(1-b)(1-c)$. Prove that $abc \leq \dfrac{1}{8}$.

Solution 1:

We have $\sqrt[3]{abc} = \sqrt[3]{(1-a)(1-b)(1-c)} \leq \dfrac{1-a+1-b+1-c}{3} = 1 - \dfrac{a+b+c}{3} \leq 1-\sqrt[3]{abc}$, hence $\sqrt[3]{abc} \leq \dfrac{1}{2}$, i.e. $abc \leq \dfrac{1}{8}$. Equality holds if and only if $a = b = c = \dfrac{1}{2}$.

Solution 2:
The inequality to be proven is equivalent to $(abc)^2 \leq \dfrac{1}{64}$, i.e. to $abc(1-a)(1-b)(1-c) \leq \dfrac{1}{64}$. It follows by multiplying the inequalities $0 < a(1-a) \leq \dfrac{1}{4}$, $0 < b(1-b) \leq \dfrac{1}{4}$, and $0 < c(1-c) \leq \dfrac{1}{4}$.
Or, alternatively, as *Ervin Macić* points out, from the AM-GM inequality we get $\sqrt[6]{abc(1-a)(1-b)(1-c)} \leq \dfrac{a+b+c+1-a+1-b+1-c}{6} = \dfrac{1}{2}$.

Solution 3:
Putting $x = \dfrac{a}{1-a}$, $y = \dfrac{b}{1-b}$, $z = \dfrac{c}{1-c}$, we have $x, y, z > 0$ with $xyz = 1$, while $a = \dfrac{x}{1+x}$, $b = \dfrac{y}{1+y}$, $c = \dfrac{z}{1+z}$, therefore the required inequality comes down to $(1+x)(1+y)(1+z) \geq 8xyz$ if $x, y, z > 0$ satisfy $xyz = 1$. But $1+x \geq 2\sqrt{x}$, $1+y \geq 2\sqrt{y}$, and $1+z \geq 2\sqrt{z}$, hence $(1+x)(1+y)(1+z) \geq 8\sqrt{xyz} = 8xyz$.
Remark: By putting, as usual, $x = \frac{m}{n}$, $y = \frac{n}{p}$, $z = \frac{p}{m}$, the inequality $(1+x)(1+y)(1+z) \geq 8xyz$ reduces to a well-known one: $(m+n)(n+p)(p+m) \geq 8mnp$ (which can be proven like the above one). This inequality is sometimes called Cesàro's inequality. Using it, it is easy to prove Euler's inequality $(R \geq 2r)$.

Solution 4: (*Ervin Macić*)
Since $a, b, cin(0,1)$, hence $x^{1/2} < x^{1/3}$, we have $\sqrt[3]{abc} > \sqrt{abc}$ or $\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)}$. We can prove $\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1$ by AM-GM, and now, since $\sqrt[3]{abc} = \sqrt[3]{(1-a)(1-b)(1-c)}$, we get that $2\sqrt[3]{abc} \leq 1$, hence the claim $abc \leq \frac{1}{8}$.