Juniors

Problem 1

Find all the pairs (n, m) of positive integers which fulfil simultaneously the conditions:

i) the number n is composite;

ii) if the numbers $d_1, d_2, \ldots, d_k, k \in \mathbb{N}^*$ are all the proper divisors of n, then the numbers $d_1 + 1, d_2 + 1, \ldots, d_k + 1$ are all the proper divisors of m.

Problem 2

Let ABC be a triangle such that in its interior there exists a point D with $\angle DAC = \angle DCA = 30^\circ$ and $\angle DBA = 60^\circ$. Denote E the midpoint of the segment BC, and take F on the segment AC so that $AF = 2FC$. Prove that $DE \perp EF$.

Problem 3

Find all the positive integers n with the property:

> *there exists an integer $k \geqslant 2$ and the positive rational numbers $a_1, a_2, \ldots, a_k$ such that $a_1 + a_2 + \ldots + a_k = a_1 a_2 \ldots a_k = n$.*

Problem 4

Let M be the set of positive odd integers. For every positive integer n, denote $A(n)$ the number of the subsets of M whose sum of elements equals n. For instance, $A(9) = 2$, because there are exactly two subsets of M with the sum of their elements equal to 9: $\{9\}$ and $\{1, 3, 5\}$.

a) Prove that $A(n) \leqslant A(n+1)$ for every integer $n \geqslant 2$.

b) Find all the integers $n \geqslant 2$ such that $A(n) = A(n+1)$.