## Solutions

**Problem 1.** Find all the pairs $(n, m)$ of positive integers which fulfil simultaneously the conditions:

i) the number $n$ is composite;

ii) if the numbers $d_1, d_2, \ldots, d_k, k \in \mathbb{N}^*$ are all the proper divisors of $n$, then the numbers $d_1 + 1, d_2 + 1, \ldots, d_k + 1$ are all the proper divisors of $m$.

**Solution.** *Answer*: $(n, m) \in \{(4, 9), (8, 15)\}$.

If $k = 1$, then $n = p^2$, where $p$ is a prime, and $m = (p+1)^2$, where $p+1$ is a prime. We get $p = 2, q = 3$, which yields the pair $(n, m) = (4, 9)$.

If $k \geqslant 2$, denote $d_1 < d_2 < \ldots < d_k$ all the proper divisors of $n$. Then

$$n = d_1 d_k = d_2 d_{k-1}$$

and

$$m = (d_1 + 1)(d_k + 1) = (d_2 + 1)(d_{k-1} + 1),$$

whence $d_1 + d_k = d_2 + d_{k-1}$, that is $d_1 + \frac{n}{d_1} = d_2 + \frac{n}{d_2}$, which is equivalent to

$$(d_1 - d_2)\Big(1 - \frac{n}{d_1 d_2}\Big) = 0.$$

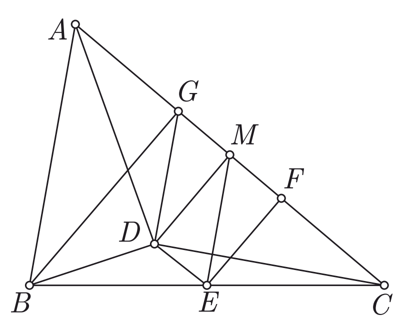
This shows that $n = d_1 d_2$, therefore $k = 2$. The possible cases are:

A) $n = d_1 d_2$, where $d_1$ and $d_2$ are different primes, hence $m = (d_1 + 1)(d_2 + 1)$. Since $d_1 + 1$ and $d_2 + 1$ must be different primes and $d_2 + 1$ is even, there is no solution in this case.

B) $n = d_1^3$, where $d_1$ is a prime, so $m = (d_1 + 1)(d_1^2 + 1)$ and $d_1 + 1$, $d_1^2 + 1$ are primes. Then $d_1 = 2$ and we get the pair $(n, m) = (8, 15)$.

**Problem 2.** Let $ABC$ be a triangle such that in its interior there exists a point $D$ with $\angle DAC = \angle DCA = 30°$ and $\angle DBA = 60°$. Denote $E$ the midpoint of the segment $BC$, and take $F$ on the segment $AC$ so that $AF = 2FC$. Prove that $DE \perp EF$.

**Solution.** Let $G$ be the midpoint of the segment $AF$ and $M$ be the midpoint of the segment $AC$. Then $M$ is the midpoint of the segment $GF$ and $DM \perp AC$.




Since $d(F, DM) = MF = \frac{GF}{2} = \frac{CF}{2} = d(F, DC)$, the ray $(DF$ is the bisector of the angle $MDC$. Then the triangle $DFG$ is equilateral.

Since $\angle DGF = \angle DBA = 60°$, the quadrilateral $ABDG$ is cyclic. Therefore, $\angle DBG = \angle DAG = 30°$ and $\angle ABG = GDA = 30°$.

The segment $EF$ is a midline of the triangle $CBG$, and the segment $EM$ is a midline of the triangle $ABC$. Hence $EF \parallel BG$ and $EM \parallel AB$, so $\angle MEF = \angle ABG = 30°$.

Since $\angle MDF = 30°$, the quadilateral $DMFE$ is cyclic, so $\angle DEF = \angle DMF = 90°$.

**Problem 3.** Find all the positive integers $n$ with the property:

> *there exists an integer $k \geqslant 2$ and the positive rational numbers $a_1, a_2, \ldots, a_k$ such that $a_1 + a_2 + \ldots + a_k = a_1 a_2 \ldots a_k = n$.*

**Solution.** *Answer*: $n \in \mathbb{N}^* \setminus \{1, 2, 3, 5\}$.

All the composite numbers are good: if $n = pq$, $p > 1, q > 1$, then we can take $a_1 = p$, $a_2 = q$ and $a_3 = a_4 \ldots = a_k = 1$, where $k = n - (p + q)$.

All the primes $n \geqslant 11$ are good: we take $a_1 = \frac{n}{2}, a_2 = \frac{1}{2}, a_3 = 4$ and $a_4 = a_5 = \ldots = a_k = 1$, where $k = \frac{n-3}{2}$.

The value $n = 7$ is good: we take $k = 3$ and $a_1 = \frac{9}{2}, a_2 = \frac{7}{6}, a_3 = \frac{4}{3}$.

Suppose now that $n \leqslant 5$, $n \neq 4$, fulfils the condition. Then the AM-GM ineqality yields $\frac{a_1+a_2+\ldots+a_k}{k} \geqslant \sqrt[k]{a_1 a_2 \ldots a_k}$, that is $n^{k-1} \geqslant k^k$.

Clearly $n = 1$ or $n = 2$ is impossible.

If $n = 3$, then $3^{k-1} < k^k$, for every $k \geqslant 2$, so this case is also impossible.

If $n = 5$, then:

- for every $k \geqslant 3$, $5^{k-1} < k^k$;
- for $k = 2$, $a_1 + a_2 = a_1 a_2 = 5$ yields irational $a_1, a_2$.

**Problem 4.** Let $M$ be the set of positive odd integers. For every positive integer $n$, denote $A(n)$ the number of the subsets of $M$ whose sum of elements equals $n$. For instance, $A(9) = 2$, because there are exactly two subsets of $M$ with the sum of their elements equal to 9: $\{9\}$ and $\{1, 3, 5\}$.

a) Prove that $A(n) \leqslant A(n + 1)$ for every integer $n \geqslant 2$.

b) Find all the integers $n \geqslant 2$ such that $A(n) = A(n + 1)$.

**Solution.** We will call $n-$set a subset of $M$ whose sum of elements is $n$.

a) The following procedure $\mathcal{P}$ associates to each $n-$set a $(n + 1)-$set, so that every two different $n-$sets have different corresponding $(n + 1)-$sets:

- if a $n-$set $S$ does not contain 1, then $S \cup \{1\}$ is a $(n + 1)-$set;
- if a $n-$set $S$ contains 1, then we replace in $S$ its largest element $2k - 1$ ($k \in \mathbb{N}$, $k \geqslant 2$) and 1 with $2k + 1$.

b) For $n \geqslant 3$, $\mathcal{P}$ leads (starting from the $n-$sets) to all the $(n + 1)-$sets which contain 1 and to the $(n + 1)-$sets which does not contain 1 and, if their largest element is $2k + 1$, then $2k - 1$ is not an element of the set.

If $n = 4k$, the $(n+1)-$set $\{5, 2k-3, 2k-1\}$ is not obtained from a $n-$set using $\mathcal{P}$, so $A(n) < A(n + 1)$ for every $n = 4k$ with $2k - 3 \geqslant 7$, that is $n \geqslant 20$.

If $n = 4k - 1$, the $(n + 1)-$set $\{2k - 1, 2k + 1\}$ is not obtained from a $n-$set using $\mathcal{P}$ for $k \geqslant 2$, that is $n \geqslant 7$.

If $n = 4k + 1$, the $(n + 1)-$set $\{3, 7, 2k - 5, 2k - 3\}$, $k \geqslant 7$ is not obtained from a $n-$set using $\mathcal{P}$, so $A(n) < A(n + 1)$ for every $n = 4k + 1$, $n \geqslant 29$.

If $n = 4k + 2$, then $\{3, 2k - 1, 2k + 1\}$, $k \geqslant 3$, is a $(n + 1)-$set which is not obtained from a $n-$set using $\mathcal{P}$, so $A(n) < A(n + 1)$ for every $n = 4k + 2 \geqslant 14$.

So, $A(n) = A(n+1)$ is possible only if $n = 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 16, 17, 21, 25$.

We get

$$A(2) = 0$$
$$A(3) = 1 = A(4) = A(5) = A(6) = A(7)$$
$$A(8) = A(9) = 2 = A(10) = A(11)$$
$$A(12) = A(13) = A(14) = 3$$
$$A(16) = A(17) = A(18) = 5$$
$$A(21) = A(22) = 8$$
$$A(25) = A(26) = 12,$$

and the above list answers the question.