Stars of Mathematics 2017, junior competition

Problem 1. How many of the first 2017 positive integers can be uniquely represented as $2^a + 2^b + 2^c$, with a, b, c non negative integers? (Two representations that only differ by the order of the terms are considered identical.)

Andrei Eckstein

Solution.
If a number can be represented as a sum of three, not necessarily distinct, powers of 2, regrouping the equal terms (if such terms exist), one obtains a sum of at most three distinct powers of 2, hence the base 2 representation of such a number has at most three digits equal to 1. Convenient numbers are those whose base 2 representation have three digits equal to 1, then the numbers of the form $2^a+1 = 2^{a-1}+2^{a-1}+1$ with $a \in \mathbb{N}$ (numbers 2^a+2^b with $a > b \geq 1$ are not convenient because $2^a + 2^b = 2^{a-1} + 2^{a-1} + 2^b = 2^{b-1} + 2^{b-1} + 2^a$); finally, the numbers $2^c = 2^{c-1} + 2^{c-1} + 2^{c-2}$ are also convenient if $c \geq 2$.
Let us count first the convenient numbers that are less than 2048. The base 2 representation of these numbers has at most 11 digits 2. There are $C_{11}^3 = 165$ numbers less than 2048 that can be written as a sum of three distinct powers of 2. There are 10 numbers of the form $2^a + 1$, $(1 \leq a \leq 10)$ and 9 of the form 2^a, $(2 \leq a \leq 10)$, hence 184 numbers in total.
The numbers from 2018 to 2047 are larger than $2^{10}+2^9+2^8$, hence their base 2 representation has more than three digits equal to 1, therefore none of these numbers is convenient.
In conclusion, there are 184 convenient numbers among the first 2017 positive integers.

Problem 2. Let x, y, z be three positive real numbers such that $x^2 + y^2 + z^2 + 3 = 2(xy + yz + zx)$. Prove that

$$\sqrt{xy} + \sqrt{yz} + \sqrt{zx} \geq 3.$$

When does the equality hold?

Vlad Robu

Solution 1.
Using the given condition, the inequality can be written equivalently $\sqrt{xy}+\sqrt{yz}+\sqrt{zx} \geq \sqrt{3(2xy + 2yz + 2zx - x^2 - y^2 - z^2)}$ or, denoting $\sqrt{x} = a$, $\sqrt{y} = b$, $\sqrt{z} = c$, we have $(ab + bc + ca)^2 \geq 3(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4)$. We thus have to prove that $3(a^4 + b^4 + c^4) + 2(a^2bc + b^2ca + c^2ab) \geq 5(a^2b^2 + b^2c^2 + c^2a^2)$.
The following inequality is well known (Schur):
$a^2(a - b)(a - c) + b^2(b - c)(b - a) + c^2(c - a)(c - b) \geq 0$.
Multiplied by 2, it becomes $2(a^4 + b^4 + c^4) + 2(a^2bc + b^2ca + c^2ab) \geq 2(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b)$.
But $2(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b) = 2ab(a^2 + b^2) + 2bc(b^2 + c^2) + 2ca(c^2 + a^2) \geq$

$4a^2b^2+4b^2c^2+4c^2a^2$ and $a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2$. Adding these three inequalities given the desired one.
Equality holds if and only if $x=y=z=1$.
Solution 2.
Rewrite $x^2+y^2+z^2+3=2(xy+yz+zx)$ as

$$(\sqrt{x}+\sqrt{y}+\sqrt{z})(\sqrt{x}+\sqrt{y}-\sqrt{z})(\sqrt{x}+\sqrt{z}-\sqrt{y})(\sqrt{y}+\sqrt{z}-\sqrt{x})=3.$$

Let $\sqrt{x}+\sqrt{y}-\sqrt{z}=2a$, $\sqrt{z}+\sqrt{y}-\sqrt{x}=2b$, $\sqrt{x}+\sqrt{z}-\sqrt{y}=2c$.
Then

$$abc(a+b+c)=\frac{3}{16}\,. \qquad (1)$$

The sum $a+b+c=\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2}$ is positive. This means that either a,b,c are all positive, or exactly two of them are negative (according to (1)). If, say, a and b are negative, then so is their sum, i.e. $a+b=\sqrt{x}<0$, which is false.
All that remains to be proven is

$$(a+b)(a+c)+(b+a)(b+c)+(c+a)(c+a)\geq 3.$$

But,

$$\sum_{cyc}(a+b)(a+c)=\sum_{cyc}a^2+3\sum_{cyc}ab\geq 4\sum_{cyc}ab\geq 4\sqrt{3abc(a+b+c)}=3,$$

which is exactly what we wanted. Equality holds if and only if $x=y=z=1$.
For the last inequality we have used $(m+n+p)^2\geq 3(mn+np+pm)$, written for $m=\sqrt{ab}$, $n=\sqrt{bc}$, $p=\sqrt{ca}$.

Problem 3. Let $P_1P_2\ldots P_n$ a regular n-gen. A frog situated at a vertex P_k ($1\leq k\leq n$) can jump to one of the vertices P_{k+2} or P_{k-3}, the indexes being considered modulo n. Determine the set of positive integers $n\geq 3$ for which the frog can make n jumps such that it visits all the vertices of the n-gon and returns to its starting vertex.

Andrei Eckstein

Solution. We prove that the convenient numbers are those that are multiples of 5 and those that are not multiples of 6.
If n has the given property, denote by a the number of jumps of type $P_k\mapsto P_{k+2}$, and let b be the number of jumps of type $P_k\mapsto P_{k-3}$. Then $a+b=n$ and $n\mid 2a-3b$, which leads to $n\mid 5a$ and $n\mid 5b$. As $0\leq a,b\leq n$, we must have either $a=0$, $b=n$, or $b=0$, $a=n$, or $5\mid n$. By only making jumps of type $P_k\mapsto P_{k+2}$, the frog can visit all the vertices if and only if n is odd; making only jumps of type $P_k\mapsto P_{k-3}$, the frog visits all the vertices if and only if $3\nmid n$. In conclusion, if n has the given property, then either it is not a

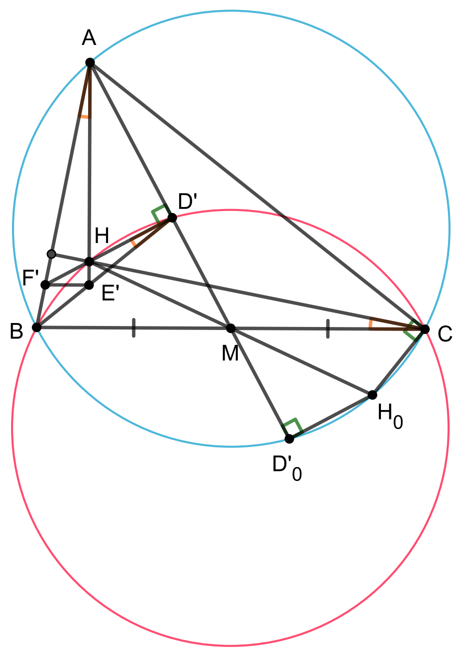
multiple of 2, or it is not a multiple of 3, or it is a multiple of 5.

Conversely, we have seen that if $6 \nmid n$, then the frog can visit all the vertices by making jumps of a single type.
All that is left is to give an example of a way to choose the frog's jumps in the case when $5 \mid n$.
There are several such examples. One of them is: the frog jumps $P_k \mapsto P_{k+2} \pmod 5$ if $5 \nmid k$ and $P_k \mapsto P_{k-3} \pmod 5$ if $5 \mid k$.

Problem 4. Let ABC be an acute triangle in which $AB < AC$. Let M be the midpoint of the side BC and consider D an arbitrary point of the line segment AM. Let E be a point of the line segment BD and consider the point F of the line AB such that lines EF and BC are parallel. If the orthocenter, H, of the triangle ABC lies at the intersection point of lines AE and DF, prove that the angle bisectors of $\angle BAC$ and $\angle BDC$ meet on the line BC.

Vlad Robu

Solution. Let D' be the orthogonal projection of H onto AM, E' the intersection point of the lines BD' and AH, and let F' be the intersection point of the lines $D'H$ and AB. We prove that $D' = D$, $E' = E$, $F' = F$, hence $HD \perp AM$.
Let H_0 and D'_0 be the reflections across the point M, of the points H and D', respectively. It is well known that H_0 is the antipode of A on the circumcircle of triangle ABC and, as $\angle H_0D'_0A = \angle HD'M = 90^\circ$, point D'_0 also lies on the circumcircle of ABC. Thus, points B, C, H, D' all lie on the reflection of this circle across M, i.e. the quadrilateral $BHD'C$ is cyclic. It follows that $\angle HD'E' = \angle HD'B = \angle HCB = 90^\circ - \angle B = \angle F'AH$, which means that the quadrilateral $AF'E'D'$ is cyclic. We obtain that $\angle F'E'A = \angle F'D'A = 90^\circ$, which means that lines $E'F'$ and BC are parallel.
If D is between A and D'), then E is between A and E', and F is between B and F' (where F is the intersection point of the lines DH and AB), hence EF can not be parallel to BC. Similarly in the case when D is between D' and M. It follows that it is necessary to have $D = D'$, and then $E = E'$, $F = F'$.
As ABD'_0C is cyclic, we have $\angle ABD'_0 = 180^\circ - \angle ACD'_0$, and therefore $\sin(\angle ABD'_0) = \sin(\angle ACD'_0)$. But since M is the midpoint of BC, triangles ABD'_0 and ACD'_0 have the same area surface, i.e. $AB \cdot BD'_0 \sin(\angle ABD'_0) = AC \cdot CD'_0 \sin(\angle ABD'_0)$, hence $AB \cdot BD'_0 = AC \cdot CD'_0$, or $AB \cdot CD = AC \cdot BD$, which means $\frac{AB}{AC} = \frac{DB}{DC}$. The converse of the Angle Bisector Theorem proves the conclusion.


A
D'
H
F'
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B
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