Mathematical Danube Competition 2017, juniors

Călăraşi, October 28, 2017

Problem 1. What is the smallest value that the sum of the digits of the number $3n^2 + n + 1$, $n \in \mathbb{N}$, can take?

Solution. (*Cristian Mangra*)
For $n = 8$ we have $3n^2 + n + 1 = 201$ whose sum of the digits is 3.
We prove that the sum of the digits of $3n^2 + n + 1$ cannot be 1 or 2. As $3n^2 + n + 1$ is odd, it cannot be written as 10^k or $2 \cdot 10^k$, $k \in \mathbb{N}$, nor can it be written as $10^k + 10^j$ with $k, j > 0$. If $3n^2 + n + 1 = 10^k + 1$, then $n(3n + 1) = 10^k$. But n and $3n + 1$ are co-prime, hence they must be 2^k and 5^k. As $n < 3n + 1$, we must have $n = 2^k$ and $3n + 1 = 5^k$, which is not possible because $5^k > 4^k > 3 \cdot 2^k + 1$ if $k \geq 2$, and $k = 1$ does not work either.

Problem 2. Let $n \geq 3$ be a positive integer. Consider an $n \times n$ square. In each cell of the square, one of the numbers from the set $M = \{1, 2, \ldots, 2n - 1\}$ is to be written. One such filling is called "*good*" if, for every index i, $1 \leq i \leq n$, row no. i and column no. i, together, contain all the elements of M.
a) Prove that there exists $n \geq 3$ for which a *good* filling exists.
b) Prove that for $n = 2017$ there is no good filling of the $n \times n$ square.

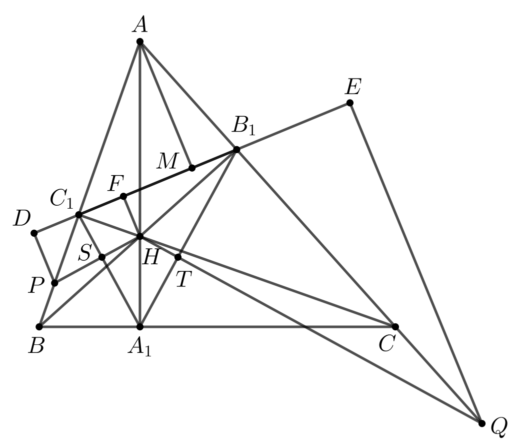
Solution.
a) For $n = 4$ the filling below is *good*.

1	2	4	5
3	1	6	4
7	5	1	2
6	7	3	1

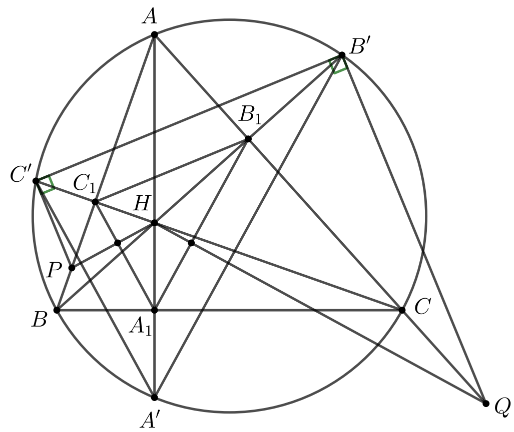
b) We prove that there is no *good* filling of an $n \times n$ square if n is odd. Assume the contrary to be true. Then, for some odd number n, we have that, for every index i, $1 \leq i \leq n$, the row no. i and the column no. i contain, together, exactly once, each element of M. If we denote by a_{ij} the number situated on row no. i and column no. j, $i, j \in \{1, 2, \ldots, n\}$, and $S_i = a_{i1} + a_{i2} + \ldots + a_{in} + a_{1i} + a_{2i} + \ldots + a_{ni}$ in which the term a_{ii} only appears once, then each element of M must appear in n such sums. But if $a_{ij} = k$, $i \neq j$, then the element k will contribute to both S_i and S_j. Thus, any number that is not situated on the main diagonal contributes to two sums. But in total, there is an odd number of sums, so each element of M must appear an odd number of times on the main diagonal. Thus, each element of M neads to appear at least once on the main diagonal, which is not possible because there are $2n - 1$ elements in M and only n places on the diagonal.

Problem 3. Consider an acute triangle ABC in which A_1, B_1, and C_1 are the feet of the altitudes dropped from A, B, and C, respectively, and H is the orthocenter. The perpendiculars dropped from H onto A_1C_1 and A_1B_1 intersect lines AB and AC at P and Q, respectively. Prove that the line perpendicular to B_1C_1 that passes through A also contains the midpoint of the line segment PQ.

Solution 1. (*Cristian Mangra*)
Point H is the incenter of triangle $A_1B_1C_1$, while A is the excenter of the same triangle, corresponding to the side B_1C_1. Let D, E, and F be the orthogonal projections of points P, Q, and H, respectively, onto the line B_1C_1. Let S be the intersection of lines PH and A_1C_1, and let T be the intersection of lines QH and A_1B_1. Then it follows that $C_1D = C_1S = C_1F$ (1), and $B_1E = B_1T = B_1F$ (2). If M is the projection of A onto B_1C_1, it follows that $FC_1 = MB_1$ (3). From (1), (2), and (3) we obtain that M is the midpoint of the line segment DE. Thus, AM is the perpendicular bisector of the line segment DE and, in the right trapezoid $DEQP$, it cuts side PQ at its midpoint.



Solution 2. (*Mircea Fianu*)
Let A', B' and C' be the reflections of point H with respect to the sides BC, CA, and AB, respectively. Triangle $A'B'C'$ is the image of triangle $A_1B_1C_1$ through a homothety centered at H, and has the same circumcircle as ABC. Moreover, $A'H$, $B'H$, and $C'H$ are the angle bisectors of triangle $A'B'C'$. Triangle QHB' is isosceles with $QH = QB'$, therefore $\angle HB'Q = \angle QHB'$. It follows that
$\angle C'B'Q = \angle HB'Q + \angle C'B'H = \angle QHB_1 + \angle C_1B_1H = \angle QHB_1 + \angle HB_1A_1 = 90^\circ$. Similarly, $\angle B'C'P = 90^\circ$. As $AB' = AC'$, the perpendicular from A to B_1C_1 is the perpendicular bisector of the line segment $B'C'$, hence parallel with QB' and PC', so it will cut the side PQ of the trapezoid $B'C'PQ$ at its midpoint.



Problem 4. Determine the triples of positive integers (x, y, z) such that $x^4+y^4 = 2z^2$ and x, y are co-prime.

Solution.
Let (x, y, z) be a solution of the problem. Then, notice that x, y are odd, hence z is also odd, and co-prime with xy. The equation can we written successively $x^8+2x^4y^4+y^8 = 4z^4$, or $(x^4 - y^4)^2 = 4z^4 - 4x^4y^4$, or $z^4 - (xy)^4 = \left(\frac{x^4-y^4}{2}\right)^2$. We prove that the equation $a^4 - b^4 = c^2 \quad (*)$, has no solutions (a, b, c), where a, b, c are positive integers and a, b are co-prime. Assume the contrary to be true. Consider $(a_0, b_0, c_0) \in \mathbb{N}^3$ the solution of the above equation with a_0 minimum. If b_0 is odd, from $b_0^4+c_0^2 = a_0^4$ we deduce that there exist positive integers m, n, $m > n$, of different parities, such that $a_0 = m^2 + n^2$, $b_0^2 = m^2 - n^2$, and $c_0 = 2mn$. It follows that $m^4 - n^4 = (a_0b_0)^2$, i.e. the triple (m, n, ab) is a solution of equation $(*)$, with $m, n, ab > 0$ and $(m, n) = 1$. This contradicts the minimality of a_0. If b is even, there exist $m, n \in \mathbb{N}$, $m > n$, such that $a^2 = m^2 + n^2$, $b^2 = 2mn$, and $c = m^2 - n^2$. We may assume that m is even, n is odd. From $b^2 = 2mn$ it follows that $2m = p^2$, $n = q^2$, i.e. $m = 2p_1^2$, $n = q^2$, with q odd, $(p_1, q) = 1$. Thus, $a^2 = (2p_1^2)^2 + (q^2)^2$, which means that there exist $r, s \in \mathbb{N}$, $r > s$, such that $a = r^2 + s^2$, $2p_1^2 = 2rs$, $q^2 = r^2 - s^2$. From $(r, s) = 1$ and $rs = p_1^2$ it follows that $r = u^2$, $s = v^2$, and $(u, v) = 1$. This means that $u^4 - v^4 = q^2$, i.e. (u, v, q) is a solution of equation $(*)$ with $u, v, q \in \mathbb{N}$, $(u, v) = 1$, and $u < a_0$, which contradicts the choice of a_0.

Getting back to the equation $z^4 - (xy)^4 = \left(\frac{x^4-y^4}{2}\right)^2$, from the above we can see that the only solutions it may have are with $x^4 - y^4 = 0$, i.e. with $x = y$. But, as $(x, y) = 1$, it follows that one must have $x = y = 1$, and finally that $x = y = z = 1$ is the only solution.