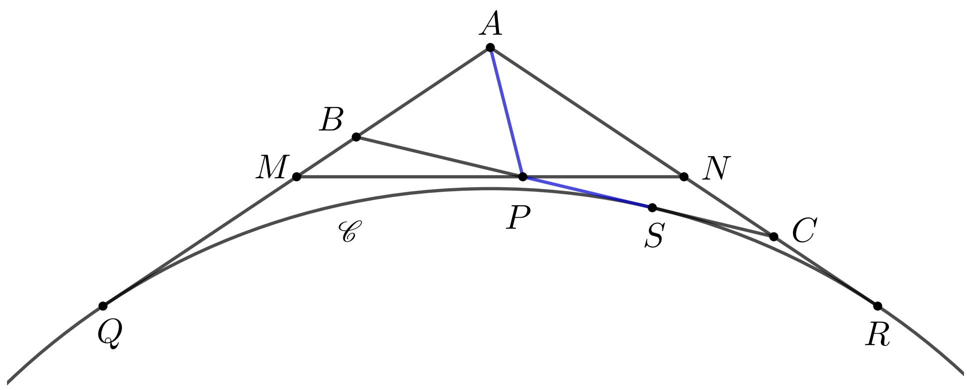
Problema săptămânii 96

Fie ABC un triunghi de perimetru 4, cu $AB \neq AC$, şi M, N două puncte pe semidreptele $(AB$, respectiv $(AC$ astfel încât $AM = AN = 1$ şi MN intersectează segmentul $[BC]$ în punctul P. Arătaţi că unul dintre triunghiurile ABP şi ACP are perimetrul 2.

Îi mulţumesc lui Cristian Mangra pentru a-mi fi semnalat problema.

Soluţia 1: (*Cristian Mangra*)

Considerăm $\mathscr{C}$, cercul exînscris triunghiului ABC tangent laturii $[BC]$. Dacă Q, R şi S sunt punctele de tangenţă ale acestui cerc cu dreptele AB, AC, respectiv BC, atunci se ştie că $AQ = AR = \dfrac{AB + BC + CA}{2} = 2$, deci M şi N sunt mijloacele segmentelor $[AQ]$, respectiv $[AR]$. Atunci M şi N au aceeaşi putere, anume 1, faţă de cercul $\mathscr{C}$ şi cercul de centru A şi rază ... 0. Aşadar MN este axa radicală a celor două cercuri. Cum P aparţine axei radicale, el are aceeaşi putere faţă de cele două cercuri, adică $PA = PS$. Dar $BQ = BS$ şi $CR = CS$, deci, dacă $P \in (BS)$, atunci $AB + BP + PA = AB + BP + PS = AB + BS = AB + BQ = AQ = 2$. Similar, dacă $P \in (CS)$, atunci perimetrul triunghiului ACP este 2.

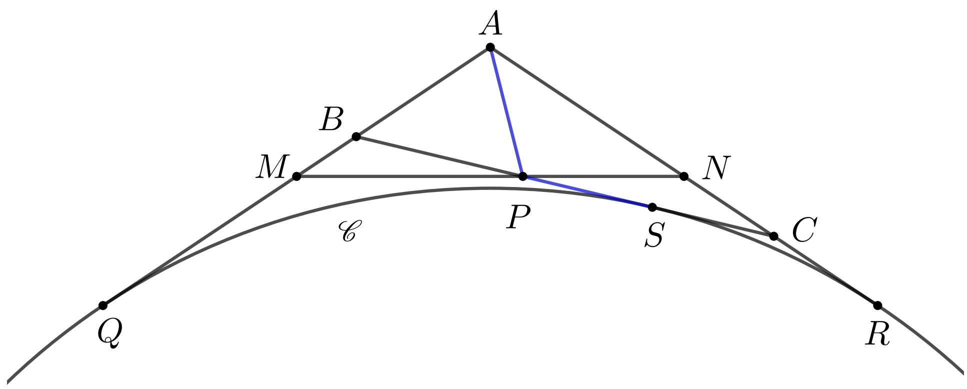


Soluţia 2: (calculatorie)

Dacă $AB < AM$, notăm $BM = x$, $CN = y$. Atunci $AB = 1 - x$, $AC = 1 + y$, deci $BC = 2 + x - y$. Din teorema sinusurilor în triunghiurile BMP şi CNP obţinem $\dfrac{BP}{BM} = \dfrac{\sin(\sphericalangle AMP)}{\sin(\sphericalangle(BPM)} = \dfrac{\sin(\sphericalangle ANP)}{\sin(\sphericalangle(CPN)} = \dfrac{\sin(\sphericalangle CNP)}{\sin(\sphericalangle(CPN)} = \dfrac{CP}{CN} = \dfrac{BM + CN}{BC} = \dfrac{x + y}{2 + x - y}$, de unde $BP = \dfrac{x(2 + x - y)}{x + y}$ şi $CP = \dfrac{y(2 + x - y)}{x + y}$. (*Lucia Rîşnoveanu* obţine aceste relaţii folosind numai teorema lui Menelaus.) Din relatia lui Stewart obţinem după calcule că $AP = \dfrac{2xy - x + y}{x + y}$, deci perimetrul triunghiului ABP este 2.

Problem of the week no. 96
Let ABC be a triangle with perimeter 4, $AB \neq AC$, ănd let M, N be two points on the rays $(AB$ and $(AC$, respectively, such that $AM = AN = 1$ and MN intersects the line segment BC at P. Prove that one of the triangles ABP and ACP has perimeter 2.

Solution 1: (*Cristian Mangra*)
Consider $\mathscr{C}$, the excircle of triangle ABC tangent to the side BC. If Q, R and S are the tangency points of this circle with lines AB, AC, and BC, respectively, then it is known that $AQ = AR = \dfrac{AB + BC + CA}{2} = 2$, hence M and N are the midpoints of the line segments AQ and $[AR]$, respectively. It follows that M and N have equal power, namely 1, with respect to the circle $\mathscr{C}$ and the circle centered at A of radius ... 0. Thus MN is the radical axis of the two circles. As P belongs to the radical axis, it has equal power with respect to the two circles, i.e. $PA = PS$. But $BQ = BS$ and $CR = CS$, hence, if P belongs to the line segment BS, then $AB + BP + PA = AB + BP + PS = AB + BS = AB + BQ = AQ = 2$. Similarly, if P belongs to the line segmentCS, then the perimeter of triangle ACP is 2.



Solution 2: (computational)
If $AB < AM$, put $BM = x$, $CN = y$. Then $AB = 1 - x$, $AC = 1 + y$, hence $BC = 2 + x - y$. From the law of the sines in triangles BMP and CNP we get $\dfrac{BP}{BM} = \dfrac{\sin(\sphericalangle AMP)}{\sin(\sphericalangle(BPM)} = \dfrac{\sin(\sphericalangle ANP)}{\sin(\sphericalangle(CPN)} = \dfrac{\sin(\sphericalangle CNP)}{\sin(\sphericalangle(CPN)} = \dfrac{CP}{CN} = \dfrac{BM + CN}{BC} = \dfrac{x + y}{2 + x - y}$, which gives $BP = \dfrac{x(2 + x - y)}{x + y}$ and $CP = \dfrac{y(2 + x - y)}{x + y}$. From Stewart's Theorem, after some computations, we obtain $AP = \dfrac{2xy - x + y}{x + y}$. Now, some

easy computations give the perimeter of triangle ABP to be 2.