ON TWO SPECIAL POINTS IN TRIANGLE

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1. Introduction

Today we will learn about two intriguing special points in triangle that seem to have many interesting properties. You will see many examples where spotting these hidden points often makes the problem very easy. These points do not seem to have a special name in the mathematical folklore. Being informal we take the liberty to call these two points as 'Humpty-dumpty points'. As you wil see, since these points are vertex dependent, we shall call them X -Humpty Dumpty points whenever they correspond to vertex X.

2. Some basic facts about Humpty-Dumpty points

2.1. Humpty point.

Definition 1. *In triangle ABC the A Humpty point P_A is defined to be a point inside triangle such that $\angle P_ABC = \angle P_AAB$ and $\angle P_ACB = \angle P_AAC$*

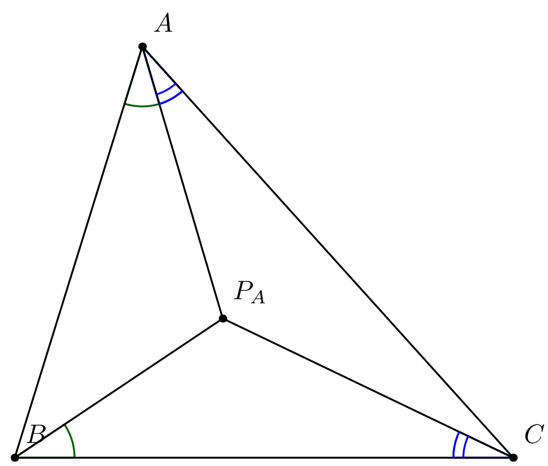

Figure 1. A-Humpty Point

Facts about P_A

1) lies on A median of ABC

2)lies on A appolonius circle of ABC that is, $\dfrac{AB}{AC} = \dfrac{P_AB}{P_AC}$

3) B, P_A, H, C are concyclic, where H is the orthocentre of ABC

4)$HP_A \perp AP_A$

2.2. **Dumpty point.**

Definition 2. *In triangle ABC, the A Dumpty point Q_A is defined to be a point inside triangle such that $\angle Q_ABA = \angle Q_AAC$ and $\angle Q_AAB = \angle Q_ACA$*

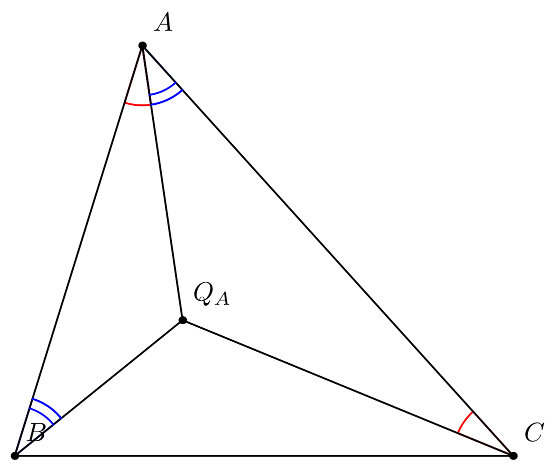

FIGURE 2. A-Dumpty point

facts about Q_A

1) lies on A symmedian

2) It is the centre of spiral symilarity sending ΔAQ_AC to ΔCQ_AB, ie sending AC to BA

3)B, Q_A, O, C are concyclic where O is circumcentre of ΔABC.

4) $OQ_A \perp AQ_A$.

Also notice that the humpty Dumpty points are isogonal conjugates of each other.

We recommend you to try to prove these facts by yourself. This will help you in befriending the humpty-dumpty points well.

If you are done, then lets move forward!

3. EXAMPLES

Problem 1 (ELMO 2014)**.** *In triangle ABC H, O are respectively the orthocentre and circumcentre respectively. Circle $\odot BOC$ intersects a circle with diameter AO at M. AM intersects $\odot BOC$ again at X. Similarly $\odot BHC$ intersects circle with diameter AH at N and AN intersects $\odot BHC$ second time at Y. Prove that $MN \parallel XY$*

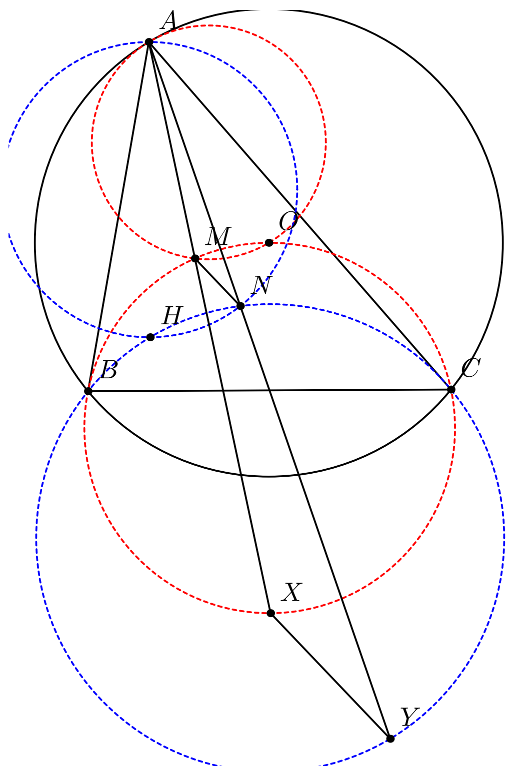

FIGURE 3. ELMO 2014

Proof. We immediately realize that N, M are A Humpty dumpty points of triangle ABC.Indeed, $HM \perp AM$ and M lies on $\odot BHC$ implies its Humpty point and similarly for the other. Using the fact that they are isogonal conjugates we chase some angles and see that $\Delta AMB \sim \Delta ACY$ and $\Delta ANB \sim \Delta ACX$. This implies

$$AM.AY = AB.AC = AN.AX$$

that is , $\dfrac{AM}{AX} = \dfrac{AN}{AY}$ and we are done. □

Problem 2 (USAMO 2008)**.** *ABC is a triangle. M, N are respectively the midpts of AB, AC. The perpendicular bisector of AB, AC meet the A median at D, E respectively. BD and CE meet at F. Prove that A, M, F, N are concyclic.*

Proof. Ah, yes F is none other than the $A-$ dumpty point of ABC. To reach the desired conclusion, see that $OF \perp AF, ON \perp AC, OM \perp AB$ directly implies A, M, O, N, F are concyclic with AO as diameter. Another way to finish would have been to see that since a spiral similiarity centered at F maps AC to BA, it also maps the midpt of AB to midpoint of AC that is, M to N. The angle of spiral similarity is $\angle AFC$ which is $180 - A$, and this must also be the $\angle MFN$ and we finish. □

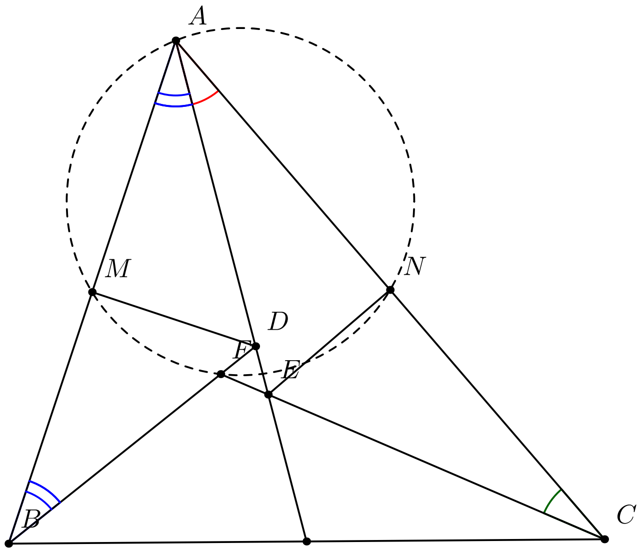

FIGURE 4. USAMO 2008

Problem 3 (USA TSTST 2015)**.** *ABC is a scalene triangle. K_a, L_a, M_a are respectively the intersections BC with the interior angle bisector,exterior angle bisector and median from A. Circle $\odot AK_aL_a$ intersects AM_a again at X_a. Similarly define X_b, X_c. Prove that the circumcentre of $X_aX_bX_c$ lies on euler line of ABC.*

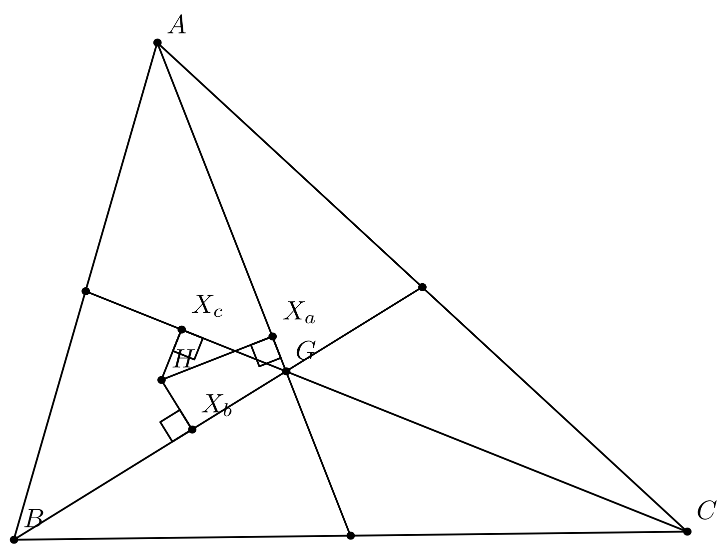

FIGURE 5. USA TSTST 2015

Proof. The problem looks really monstrous at first sight, but we will see how innocent it is. Clearly, $\odot AK_aL_a$ is the A applonius circle of ABC and so we have X_a as the A Humpty point of ABC. let us invoke another property of X_a, namely

$HX_a \perp AM_a$, similarly for b, c. what do we get? Since AM_a, BM_b, CM_c are concurrent at centroid G, which implies X_a, X_b, X_c lies on a circle with diameter HG. So the centre is midpt of HG, lies on euler line indeed. □

Problem 4 (USA TST 2005)**.** *P is a point inside triangle ABC such that $\angle PAB = \angle PBC, \angle PAC = \angle PCB$. The perpendicular bisector of AP meets BC at Q. If O is the circumcentre of ABC then prove that $\angle AQP = 2\angle OQB$*

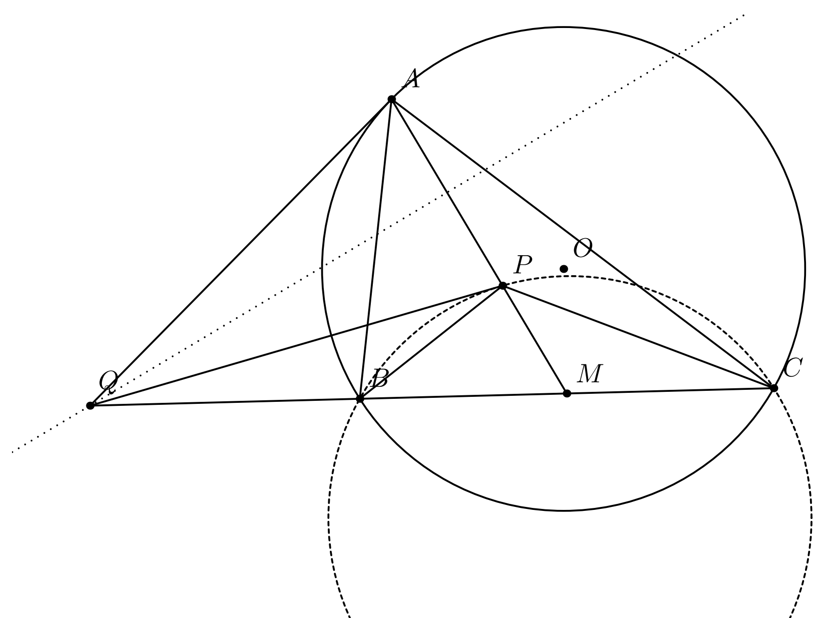

FIGURE 6. USA TST 2005

Proof. Sure enough, it shouldnt take us much time to realize that P is the A Humpty point of ABC, hence AP is A median. Let AP intersect BCat M. The condition of problem requires $90 - \frac{1}{2}\angle AQP = 90 - \angle OQB \Leftrightarrow \angle QAM = \angle QOM \Leftrightarrow AQMO$ is cyclic, that is $\angle QAO = 90$ or QA tangent to $\odot ABC$.

This suggests that we should start working by taking Q' such that $Q'A$ tangent to $\odot ABC$ and try to prove that $Q'A = Q'P$. We know that $\dfrac{AB}{AC} = \dfrac{BQ'^2}{CQ'^2}$. also we know $\dfrac{AB}{AC} = \dfrac{PB}{PC}$, from properties of Humpty point. Hence we have

$$\frac{PB}{PC} = \frac{BQ'^2}{CQ'^2}$$

. We show that this property leads to $Q'P$ tangent to $\odot PBC$. Indeed, if tangent from P to $\odot PBC$ meets BC at Q" then $\dfrac{PB}{PC} = \dfrac{BQ"^2}{CQ"^2}$ so $\dfrac{BQ"^2}{CQ"^2} = \dfrac{BQ'^2}{CQ'^2}$ which implies $Q' = Q$". Now by power of point, $Q'A^2 = Q'B.Q'C = Q'P^2$. Hence $Q'A = Q'P$ and we are done. □

4. EXCERCISES

Given are some problems for the reader to try.

Exercise 1. *The A symmedian of ABC meets its circumcircle at K. Reflection of K in BC is K'. Prove that AK' is a median.*

Exercise 2. *P is a variable point on side BC of triangle ABC. M, N are respectively on AB, AC such that $PM \parallel AC, PN \parallel AB$. Prove that as P varies on BC, $\odot AMN$ passes through a fixed point.*

Exercise 3. *M, N are points on semicircle with diameter AB and centre O. NM meets AB at X. circles $\odot MBO, \odot NAO$ meet at K. Prove that $XK \perp KO$*

Exercise 4. *Q_A is the A dumpty point of ABC. AD is altitude from A on BC. Prove that DQ_A bisects the line segment joining the midpoints of AB, AC*

Exercise 5. *P is point on A symmedian of ABC. O_1, O_2 are respectively circumcentres of APB, CAP. If O is circumcentre of ABC, prove that AO bisects O_1O_2*

Exercise 6. *AD is an altitude from A to BC in triangle ABC. A circle with centre on AD is tangent externally to $\odot BOC$ at X, where O iis circumcentre of ABC. Prove that AX is the A symmedian.*