Problema săptămânii 89

Demonstraţi că pentru orice a, b, c pozitive are loc inegalitatea

$$\frac{2ab}{a+b}+\frac{2bc}{b+c}+\frac{2ca}{c+a}\le\frac{3ab+3bc+3ca}{a+b+c}.$$

Soluţia 1: (*Vlad Vergelea*)

Avem $\frac{2ab}{a+b}\le\frac{2ab+\frac{bc+ca}{2}}{a+b+c}\Leftrightarrow 2ab(a+b+c)\le 2ab(a+b)+(a+b)^2\cdot\frac{c}{2}\Leftrightarrow 2abc\le (a+b)^2\cdot\frac{c}{2}\Leftrightarrow c(a-b)^2\ge 0$, inegalitate adevărată, cu egalitate dacă $a=b$.

Scriind şi relaţiile analoage şi adunând obţinem inegalitatea din enunţ, cu egalitate dacă $a=b=c$.

Soluţia 2: (*Andrei Mărginean*)

Înmulţind cu $a+b+c$, inegalitatea de demonstrat se rescrie echivalent $2ab+2bc+2ca+2abc\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)\le 3ab+3bc+3ca$, adică, împărţind cu abc, $\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}\le\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

Ultima inegalitate rezultă din $\frac{1}{a+b}\le\frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right)$ şi analoagele.

Egalitate avem dacă şi numai dacă $a=b=c$.

Soluţia 3: (*David Andrei Anghel*)

Inegalitatea se poate scrie echivalent $\frac{a+b}{2}-\frac{2ab}{a+b}+\frac{b+c}{2}-\frac{2bc}{b+c}+\frac{c+a}{2}-\frac{2ca}{c+a}\ge a+b+c-\frac{3ab+3bc+3ca}{a+b+c}$, adică $\frac{(a-b)^2}{2(a+b)}+\frac{(b-c)^2}{2(b+c)}+\frac{(c-a)^2}{2(c+a)}\ge\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a+b+c)}$, inegalitate care rezultă din adunarea inegalităţii evidente $\frac{(a-b)^2}{2(a+b)}\ge\frac{(a-b)^2}{2(a+b+c)}$ cu analoagele ei. Egalitate avem dacă şi numai dacă $a=b=c$.

Soluţia 4:

Eliminând numitorii şi reducând termenii asemenea se ajunge la $a^3b^2+a^3c^2+b^3a^2+b^3c^2+c^3a^2+c^3b^2\ge 2(a^2b^2c+b^2c^2a+c^2a^2b)$, adică la $[3,2,0]\ge[2,2,1]$ care rezultă direct din inegalitatea Muirhead.

Soluţia 5: (*Marian Cucoaneş*)

Avem $\frac{3ab+3bc+3ca}{a+b+c}-\frac{2ab}{a+b}-\frac{2bc}{b+c}-\frac{2ca}{c+a}=\frac{a^2(b-c)^2}{(a+b)(a+c)(a+b+c)}+\frac{b^2(c-a)^2}{(b+a)(b+c)(a+b+c)}+\frac{c^2(a-b)^2}{(a+c)(b+c)(a+b+c)}\ge 0$, cu egalitate dacă şi numai dacă $a = b = c$.

Generalizare: (*Marian Cucoaneş*)
Fie $n \in \mathbb{N}$, $n \geq 2$ şi $x_1, x_2, \ldots, x_n > 0$. Notăm $S = x_1 + x_2 + \ldots + x_n$, $P = x_1 \cdot x_2 \cdot \ldots \cdot x_n$, $S_i = S - x_i$, $P_i = \dfrac{P}{x_i}$, $\forall\, i = \overline{1, n}$. Atunci

$$(n-1)\sum_{i=1}^{n}\frac{P_i}{S_i} \leq \frac{n}{S} \cdot \sum_{i=1}^{n} P_i.$$

Remarcă: (*Titu Zvonaru*)
Dacă în membrul stâng, în locul mediilor armonice ale numerelor, a şi b, b şi c, c şi a punem mediile aritmetice, sensul inegalităţii se schimbă, adică avem

$$\frac{2ab}{a+b} + \frac{2bc}{b+c} + \frac{2ca}{c+a} \leq \frac{3ab+3bc+3ca}{a+b+c} \leq \frac{a+b}{2} + \frac{b+c}{2} + \frac{c+a}{2}.$$

Dacă punem media geometrică obţinem o inegalitate care nu are loc în niciunul din sensuri: pentru $a = b = 1$, $c = 2$ avem

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} > \frac{3ab+3bc+3ca}{a+b+c},$$

iar pentru $a = 1$, $b = c = 5$ avem

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} < \frac{3ab+3bc+3ca}{a+b+c}.$$

Problem of the week no. 89
Prove that the inequality

$$\frac{2ab}{a+b} + \frac{2bc}{b+c} + \frac{2ca}{c+a} \leq \frac{3ab+3bc+3ca}{a+b+c}$$

holds for all positive real numbers a, b, c.
Solution 1: (*Vlad Vergelea*)
We have $\dfrac{2ab}{a+b} \leq \dfrac{2ab + \dfrac{bc+ca}{2}}{a+b+c} \Leftrightarrow 2ab(a+b+c) \leq 2ab(a+b) + (a+b)^2 \cdot \dfrac{c}{2} \Leftrightarrow$ $2abc \leq (a+b)^2 \cdot \dfrac{c}{2} \Leftrightarrow c(a-b)^2 \geq 0$, which is true. Equality holds when $a = b$.
Adding this inequality with two similar ones yields the desired inequality, which is satisfied with equality when $a = b = c$.

Solution 2: (*Andrei Mărginean*)
Multiplying by $a+b+c$, the inequality can be rewritten equivalently $2ab + 2bc + 2ca + 2abc\left(\dfrac{1}{a+b} + \dfrac{1}{b+c} + \dfrac{1}{c+a}\right) \leq 3ab + 3bc + 3ca$, i.e., dividing by abc, $\dfrac{2}{a+b} + \dfrac{2}{b+c} + \dfrac{2}{c+a} \leq \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.
The last inequality follows from $\dfrac{1}{a+b} \leq \dfrac{1}{4}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$ and its analogues.

Equality holds if and only if $a = b = c$.

Solution 3: (*David Andrei Anghel*)
The inequality can be written

$$\frac{a+b}{2} - \frac{2ab}{a+b} + \frac{b+c}{2} - \frac{2bc}{b+c} + \frac{c+a}{2} - \frac{2ca}{c+a} \geq a+b+c - \frac{3ab+3bc+3ca}{a+b+c},$$

i.e.

$$\frac{(a-b)^2}{2(a+b)} + \frac{(b-c)^2}{2(b+c)} + \frac{(c-a)^2}{2(c+a)} \geq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a+b+c)},$$

which follows from adding $\dfrac{(a-b)^2}{2(a+b)} \geq \dfrac{(a-b)^2}{2(a+b+c)}$ with its analogues.
Equality holds if and only if $a = b = c$.

Solution 4:
Multiplying out reduces the inequality to $a^3b^2 + a^3c^2 + b^3a^2 + b^3c^2 + c^3a^2 + c^3b^2 \geq 2(a^2b^2c + b^2c^2a + c^2a^2b)$, i.e. $[3,2,0] \geq [2,2,1]$ which follows directly from Muirhead's inequality.

Solution 5: (*Marian Cucoaneş*)
We have $\dfrac{3ab+3bc+3ca}{a+b+c} - \dfrac{2ab}{a+b} - \dfrac{2bc}{b+c} - \dfrac{2ca}{c+a} = \dfrac{a^2(b-c)^2}{(a+b)(a+c)(a+b+c)} + \dfrac{b^2(c-a)^2}{(b+a)(b+c)(a+b+c)} + \dfrac{c^2(a-b)^2}{(a+c)(b+c)(a+b+c)} \geq 0$, with equality if and only if $a = b = c$.

More generally: (*Marian Cucoaneş*)
Let $n \in \mathbb{N}$, $n \geq 2$ and $x_1, x_2, \ldots, x_n > 0$. Let $S = x_1 + x_2 + \ldots + x_n$, $P = x_1 \cdot x_2 \cdot \ldots \cdot x_n$, $S_i = S - x_i$, $P_i = \dfrac{P}{x_i}$, $\forall\, i = \overline{1,n}$. Prove that

$$(n-1)\sum_{i=1}^{n} \frac{P_i}{S_i} \leq \frac{n}{S} \cdot \sum_{i=1}^{n} P_i.$$