XIV GEOMETRICAL OLYMPIAD IN HONOUR OF I.F.SHARYGIN
The correspondence round

Below is the list of problems for the first (correspondence) round of the XIV Sharygin Geometrical Olympiad.

The olympiad is intended for high-school students of four eldest grades. In Russian school, these are 8-11. In the list below, each problem is indicated by the numbers of Russian school grades, for which it is intended. Foreign students of the last grade have to solve the problems for 11th grade, students of the preceding grade solve the problems for 10th grade etc. However, the participants may solve problems for elder grades as well (solutions of problems for younger grades will not be considered).

A complete solution of each problem costs 7 points. A partial solution costs from 1 to 6 points. A text without significant advancement costs 0 points. The result of a participant is the sum of all obtained marks.

Please include the solution of each problem in a separate file. First write down the statement of the problem, and then the solution. Present your solutions in detail, including all necessary arguments and calculations. Provide all necessary figures of sufficient size. If a problem has an explicit answer, this answer must be presented distinctly. Please, be accurate to provide good understanding and correct estimating of your work !

If your solution depends on some well-known theorems from standard textbooks, you may simply refer to them instead of providing their proofs. However, any fact not from the standard curriculum should be either proved or properly referred (with an indication of the source).

You may note the problems which you liked most (this is not obligatory). Your opinion is interesting for the Jury.

The solutions for the problems (in Russian or in English) must be delivered not before January 8, 2018 and not later than on April 1, 2018. To upload your work, enter the site **https://contest.yandex.ru/geomshar/**, indicate the language (English) in the right upper part of the page, press "Registration"in the left upper part, and follow the instructions.

Attention:

1. The solution of each problem must be contained in a *separate* pdf, doc or jpg file. We recommend to prepare the paper using computer or to scan it rather than to photograph it. *In the last two cases, please check readability of the file before uploading.*

2. If you upload the solution of some problem more than once then only the last version is retained in the checking system. Thus if you need to change something in your solution then you have to upload the whole solution again.

If you have any technical problems with uploading of the work, apply to **geomshar@yandex.ru (DON'T SEND your work to this address)**.

The final round will be held in July–August 2018 in Moscow region. The winners of the correspondence round are invited to it if they don't graduate from school before. (For instance, if the last grade is 12 then we invite winners from 9–11 grades, and from 12 grade if they finish their education later.) The graduates, winners of the correspondence round, will be awarded by diplomas of the Olympiad. The list of the winners will be published on **www.geometry.ru** at the end of May 2018 at latest. If you want to know your detailed results, please use e-mail **geomshar@yandex.ru**.

1. (grade 8) Three circles lie inside a square. Each of them touches externally two remaining

circles. Also each circle touches two sides of the square. Prove that two of these circles are congruent.

2. (grade 8) A cyclic quadrilateral $ABCD$ is given. The lines AB and DC meet at point E, and the lines BC and AD meet at point F. Let I be the incenter of triangle AED, and a ray with origin F be perpendicular to the bisector of angle AID. In which ratio this ray dissects the angle AFB?

3. (grade 8) Let AL be the bisector of triangle ABC, D be its midpoint, and E be the projection of D to AB. It is known that $AC = 3AE$. Prove that CEL is an isosceles triangle.

4. (grade 8) Let $ABCD$ be a cyclic quadrilateral. A point P moves along the arc AD which does not contain B and C. A fixed line l, perpendicular to BC, meets the rays BP, CP at points B_0, C_0 respectively. Prove that the tangent at P to the circumcircle of triangle PB_0C_0 passes through some fixed point.

5. (grades 8–9) The vertex C of equilateral triangles ABC and CDE lies on the segment AE, and the vertices B and D lie on the same side with respect to this segment. The circumcircles of these triangles centered at O_1 and O_2 meet for the second time at point F. The lines O_1O_2 and AD meet at point K. Prove that $AK = BF$.

6. (grades 8–9) Let CH be the altitude of a right-angled triangle ABC ($\angle C = 90^\circ$) with $BC = 2AC$. Let O_1, O_2 and O be the incenters of triangles ACH, BCH and ABC respectively, and H_1, H_2, H_0 be the projections of O_1, O_2, O respectively to AB. Prove that $H_1H = HH_0 = H_0H_2$.

7. (grades 8–9) Let E be a common point of circles w_1 and w_2. Let AB be a common tangent to these circles, and CD be a line parallel to AB, such that A and C lie on w_1, B and D lie on w_2. The circles ABE and CDE meet for the second time at point F. Prove that F bisects one of arcs CD of circle CDE.

8. (grades 8–9) Restore a triangle ABC by the Nagel point, the vertex B and the foot of the altitude from this vertex.

9. (grades 8–9) A square is inscribed into an acute-angled triangle: two vertices of this square lie on the same side of the triangle and two remaining vertices lies on two remaining sides. Two similar squares are constructed for the remaining sides. Prove that three segments congruent to the sides of these squares can be the sides of an acute-angled triangle.

10. (grades 8–9) In the plane, 2018 points are given such that all distances between them are different. For each point, mark the closest one of the remaining points. What is the minimal number of marked points?

11. (grades 8–9) Let I be the incenter of a nonisosceles triangle ABC. Prove that there exists a unique pair of points M, N lying on the sides AC, BC respectively, such that $\angle AIM = \angle BIN$ and $MN \parallel AB$.

12. (grades 8–9) Let BD be the external bisector of a triangle ABC with $AB > BC$; K and K_1 be the touching points of side AC with the incicrle and the excircle centered at I and

I_1 respectively. The lines BK and DI_1 meet at point X, and the lines BK_1 and DI meet at point Y. Prove that $XY \perp AC$.

13. (grades 9–11) Let $ABCD$ be a cyclic quadrilateral, and M, N be the midpoints of arcs AB and CD respectively. Prove that MN bisects the segment between the incenters of triangles ABC and ADC.

14. (grades 9–11) Let ABC be a right-angled triangle with $\angle C = 90°$, K, L, M be the midpoints of sides AB, BC, CA respectively, and N be a point of side AB. The line CN meets KM and KL at points P and Q respectively. Points S, T lying on AC and BC respectively are such that $APQS$ and $BPQT$ are cyclic quadrilaterals. Prove that

    a) if CN is a bisector, then CN, ML and ST concur;

    b) if CN is an altitude, then ST bisects ML.

15. (grades 9–11) The altitudes AH_1, BH_2, CH_3 of an acute-angled triangle ABC meet at point H. Points P and Q are the reflections of H_2 and H_3 with respect to H. The circumcircle of triangle PH_1Q meets for the second time BH_2 and CH_3 at points R and S. Prove that RS is a medial line of triangle ABC.

16. (grades 9–11) Let ABC be a triangle with $AB < BC$. The bisector of angle C meets the line parallel to AC and passing through B, at point P. The tangent at B to the circumcircle of ABC meets this bisector at point R. Let R' be the reflection of R with respect to AB. Prove that $\angle R'PB = \angle RPA$.

17. (grades 10–11) Let each of circles α, β, γ touches two remaining circles externally, and all of them touche a circle Ω internally at points A_1, B_1, C_1 respectively. The common internal tangent to α and β meets the arc A_1B_1 not containing C_1 at point C_2. Points A_2, B_2 are defined similarly. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

18. (grades 10–11) Let C_1, A_1, B_1 be points on sides AB, BC, CA of triangle ABC, such that AA_1, BB_1, CC_1 concur. The rays B_1A_1 and B_1C_1 meet the circumcircle of the triangle at points A_2 and C_2 respectively. Prove that A, C, the common point of A_2C_2 and BB_1 and the midpoint of A_2C_2 are concyclic.

19. (grades 10–11) Let a triangle ABC be given. On a ruler three segment congruent to the sides of this triangle are marked. Using this ruler construct the orthocenter of the triangle formed by the tangency points of the sides of ABC with its incircle.

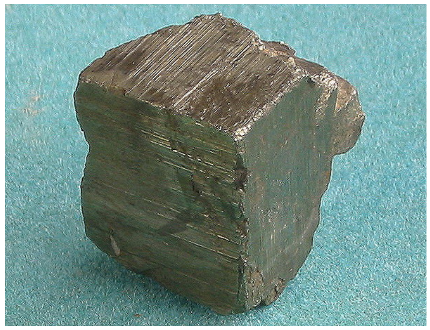
20. (grades 10–11) Let the incircle of a nonisosceles triangle ABC touch AB, AC and BC at points D, E and F respectively. The corresponding excircle touches the side BC at point N. Let T be the common point of AN and the incircle, closest to N, and K be the common point of DE and FT. Prove that $AK || BC$.

21. (grades 10–11) In the plane a line l and a point A outside it are given. Find the locus of the incenters of acute-angled triangles having a vertex A and an opposite side lying on l.

22. (grades 10–11) Six circles of unit radius lie in the plane so that the distance between the centers of any two of them is greater than d. What is the least value of d such that there always exists a straight line which does not intersect any of the circles and separates the circles into two groups of three?

23. (grades 10–11) The plane is divided into convex heptagons with diameters less than 1. Prove that an arbitrary disc with radius 200 intersects most than a billion of them.

24. (grades 10–11) A crystal of pyrite is a parallelepiped with dashed faces.



The dashes on any two adjacent faces are perpendicular. Does there exist a convex polytope with the number of faces not equal to 6, such that its faces can be dashed in such a manner?