Another solution to problem 2

When I first saw the problem I thought that the key to solving it was to exploit the fact that the numbers were distinct integers.
The inequality can be written equivalently $(x+y+z)(xy+yz+zx) \geq 9xyz + 2(x+y+z)$. If x, y, z were just some positive real numbers, not subjected to any constraint, we would only have $(x+y+z)(xy+yz+zx) \geq 9xyz$ (with equality when $x = y = z$). I thought that maybe I should consider the proof of this last equality and adapt it somehow to our context. We have

$$(x+y+z)(xy+yz+zx) \geq 9xyz \Leftrightarrow x(y-z)^2 + y(z-x)^2 + z(x-y)^2 \geq 0.$$

So what we actually need to prove is $x(y-z)^2+y(z-x)^2+z(x-y)^2 \geq 2(x+y+z)$ whenever x, y, z are distinct positive integers.
It is clear that this is true in the case when $(y-z)^2 \geq 2$, $(z-x)^2 \geq 2$, $(x-y)^2 \geq 2$, i.e. when no two of the numbers are consecutive.
This motivates considering cases when some of the numbers are consecutive.

A slightly different approach is to profit of the simmetry of the inequality. We can assume a certain order between the variables, say $x < y < z$.
A standard continuation, one that allows exploiting the fact that the variables differ by at least 1, is to put $a = y - x$ and $b = z - y$. We have a and b positive integers and $y = x + a$, $z = x + a + b$. In these terms, the last inequality comes to $xb^2 + (x+a)(a+b)^2 + (x+a+b)a^2 \geq 2(3x+2a+b)$, i.e. to $x\big(a^2+b^2+(a+b)^2-6\big) + a\big((a+b)^2+a^2-4\big) + b(a^2-2) \geq 0$. All the terms are non-negative and the last one is positive if $a > 1$, so the inequality is fulfilled with no equality cases.
It remains to study the case when $a = 1$.
In this case, the above inequality becomes $x\big(2b^2+2b-4\big) + (b^2+b-2) \geq 0$. Both terms are non-negative and equal to 0 if and only if $b = 1$, so the inequality is proven. We have equality if and only if $b = 1$ (and $a = 1$), i.e. when the three numbers are consecutive.