Problem 1. Find all the positive integers x, y, z, t such that $2^x \cdot 3^y + 5^z = 7^t$.

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Solution. Reducing modulo 3 we get $5^z \equiv 1$, therefore z is even, $z = 2c$, $c \in \mathbb{N}$

Next we prove that t is even. Obviously, $t \geq 2$. Let us suppose that t is odd, $t = 2d + 1$, $d \in \mathbb{N}$. The equation becomes $2^x \cdot 3^y + 25^c = 7 \cdot 49^d$.
If $x \geq 2$, reducing modulo 4, we get $1 \equiv 3$, contradiction.
For $x = 1$, we have $2 \cdot 3^y + 25^c = 7 \cdot 49^d$, and, reducing modulo 24, we obtain $2 \cdot 3^y + 1 \equiv 7 \Rightarrow 24 \mid 2\left(3^y - 3\right)$, i.e. $4 \mid 3^{y-1} - 1$, which means that $y - 1$ is even. Then, $y = 2b + 1$, $b \in \mathbb{N}$.

We obtain $6 \cdot 9^b + 25^c = 7 \cdot 49^d$, and, reducing modulo 5, we get $(-1)^b \equiv 2 \cdot (-1)^d$, which is false, for all $b, d \in \mathbb{N}$. Hence t is even, $t = 2d$, $d \in \mathbb{N}$.

The equation can be written as $2^x \cdot 3^y + 25^c = 49^d \Leftrightarrow 2^x \cdot 3^y = \left(7^d - 5^c\right)\left(7^d + 5^c\right)$.
As $\gcd\left(7^d - 5^c, 7^d + 5^c\right) = 2$ and $7^c + 5^c > 2$, there are three possible situations:

$$(1) \begin{cases} 7^d - 5^c = 2^{x-1} \\ 7^d + 5^c = 2 \cdot 3^y \end{cases}; \qquad (2) \begin{cases} 7^d - 5^c = 2 \cdot 3^y \\ 7^d + 5^c = 2^{x-1} \end{cases}; \qquad (3) \begin{cases} 7^d - 5^c = 2 \\ 7^d + 5^c = 2^{x-1} \cdot 3^y \end{cases}.$$

Case (1). We have $7^d = 2^{x-2} + 3^y$ and, reducing modulo 3, we get $2^{x-2} \equiv 1 \pmod 3$, hence $x - 2$ is even, i.e. $x = 2a + 2$, where $a \in \mathbb{N}$, since $a = 0$ would mean $3^y + 1 = 7^d$, which is impossible (even = odd).

We obtain $7^d - 5^c = 2 \cdot 4^a \overset{\text{mod } 4}{\Longrightarrow} 7^d \equiv 1 \pmod 4 \Rightarrow d = 2e$, $e \in \mathbb{N}$. Then $49^e - 5^c = 2 \cdot 4^k \overset{\text{mod } 8}{\Longrightarrow} 5^c \equiv 1 \pmod 8 \Rightarrow c = 2f$, $f \in \mathbb{N}$. We obtain $49^e - 25^f = 2 \cdot 4^a \overset{\text{mod } 3}{\Longrightarrow} 0 \equiv 2 \pmod 3$, false. In conclusion, in this case there are no solutions to the equation.

Case (2). From $2^{x-1} = 7^d + 5^c \geq 12$, we obtain $x \geq 5$. Then $7^d + 5^c \equiv 0 \pmod 4$, i.e. $3^d + 1 \equiv 0 \pmod 4$, hence d is odd. As $7^d = 5^c + 2 \cdot 3^y \geq 11$, we get $d \geq 2$, hence $d = 2e + 1$, $e \in \mathbb{N}$.
As in the previous case, from $7^d = 2^{x-2} + 3^y$, reducing modulo 3, we obtain $x = 2a + 2$, with $a \geq 2$ (because $x \geq 5$). We get $7^d = 4^a + 3^y$, i.e. $7 \cdot 49^e = 4^a + 3^y$, hence, reducingmodulo 8, we obtain $7 \equiv 3^y$, which is false, because 3^y is congruent mod 8 either to 1 (if y is even) or to 3 (if y is odd). In conclusion, in this case there are no solutions to the equation.

Case (3). From $7^d = 5^c + 2$, it follows that the last digit of 7^d is 7, hence $d = 4k + 1$, $k \in \mathbb{N}$.
If $c \geq 2$, from $7^{4k+1} = 5^c + 2$, reducing modulo 25, we obtain $7 \equiv 2 \pmod{25}$, which is false.
For $c = 1$ we get $d = 1$, and the solution $x = 3$, $y = 1$, $z = t = 2$.

Problem 2. Find the largest positive integer n for which the inequality

$$\frac{a+b+c}{abc+1}+\sqrt[n]{abc}\leq\frac{5}{2}$$

holds for all $a,b,c\in[0,1]$.

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Solution. Let $E(n)=\dfrac{a+b+c}{abc+1}+\sqrt[n]{abc}$. Then $E(m)-E(n)=\sqrt[m]{abc}-\sqrt[n]{abc}$.

As $abc\leq 1\Leftrightarrow E(m)\geq E(n)\ \forall m\geq n$, it is sufficient to find n such that $\dfrac{a+b+c}{abc+1}+\sqrt[n]{abc}\leq\dfrac{5}{2}$ $\forall a,b,c\in[0,1]$ and $\exists a,b,c$ such that $\dfrac{a+b+c}{abc+1}+\sqrt[n+1]{abc}>\dfrac{5}{2}$.

Let us determine an upper bound for n, by plugging some particular values into the inequality.

For $(1,1,c)$ we obtain $\dfrac{c+2}{c+1}+\sqrt[n]{c}\leq\dfrac{5}{2}$, $\forall c\in[0,1]\Leftrightarrow\dfrac{1}{c+1}+\sqrt[n]{c}\leq\dfrac{3}{2}$. Let $\sqrt[n]{c}=x$. It is obvious that $\forall x\in[0;1]$ can be written as $\sqrt[n]{c}$ for a certain $c\in[0;1]$.

The inequality becomes $\dfrac{1}{x^n+1}+x\leq\dfrac{3}{2}\Leftrightarrow 2+2x^{n+1}+2x\leq 3x^n+3\Leftrightarrow 3x^n+1\geq 2x^{n+1}+2x\Leftrightarrow$ $2x^n(1-x)+(1-x)+(x-1)(x^{n-1}+\cdots+x)\geq 0\Leftrightarrow(1-x)[2x^n+1-(x^{n-1}+x^{n-2}+\ldots+x)]\geq 0\quad(*)$.

For $n=4$ we have $(1-x)(2x^n+1-x^{n-2}-\ldots-x)=(1-x)(2x^4+1-x^3-x^2-x)=(1-x)(x-1)(2x^3+x^2-1)=-(1-x)^2(2x^3+x^2-1)$.
For $x=0.9$ the inequality $(*)$ is no longer true, and, according to the remark from the beginning of the proof, the inequality in not fulfilled if $n\geq 4$.

Now, we shall prove that for $n=3$ the inequality holds. We shall use the following result:

For all $a,b,c\in[0;1]:\ abc+2\geq a+b+c$.
Proof: $(a-1)(b-1)\geq 0\Leftrightarrow ab+1\geq a+b\Leftrightarrow 1\geq a+b-ab$.
$(ab-1)(c-1)\geq 0\Leftrightarrow abc+1\geq ab+c$.
Adding these two inequalities we obtain $a,b,c\in[0;1]\Rightarrow abc+2\geq a+b+c$.

The inequality reduces to $\dfrac{abc+2}{abc+1}+\sqrt[3]{abc}\leq\dfrac{5}{2}\Leftrightarrow\dfrac{1}{abc+1}+\sqrt[3]{abc}\leq\dfrac{3}{2}$. Denoting $\sqrt[3]{abc}=y\in[0;1]$ the inequality reduces to: $\dfrac{1}{y^3+1}+y\leq\dfrac{3}{2}\Leftrightarrow 2+2y^4+2y\leq 3y^3+3\Leftrightarrow -2y^4+3y^3-2y+1\geq 0\Leftrightarrow$ $2y^3(1-y)+(y-1)y(y+1)+(1-y)\geq 0\Leftrightarrow(1-y)(2y^3+1-y^2-y)\geq 0$. The last inequality is obvious because $y^3+1\geq y^2+y\Leftrightarrow(y-1)^2(y+1)\geq 0$ and $y^3\geq 0$.

In conclusion, $n=3$.

Problem 3. Let $MNPQ$ be a square of side length 1, and A, B, C, D points on the sides MN, NP, PQ, and QM respectively such that $AC \cdot BD = \frac{5}{4}$. Can the set $\{AB, BC, CD, DA\}$ be partitioned into two subsets S_1 and S_2 of two elements each such that both the sum of the elements of S_1 and the sum of the elements of S_2 are positive integers?

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Solution. The answer is negative.
Suppose such a partitioning was possible. Then $AB + BC + CD + DA \in \mathbb{N}$.
But $(AB + BC) + (CD + DA) > AC + AC \geq 2$, hence $AB + BC + CD + DA > 2$.
On the other hand, $AB + BC + CD + DA < (AN + NB) + (BP + PC) + (CQ + QD) + (DM + MA) = 4$, hence $AB + BC + CD + DA = 3$.
Obviously the sums of the elements of S_1 and S_2 must be 1 and 2. Without loss of generality, we may assume that the sum of the elements of S_1 is 1 and the sum of the elements of S_2 is 2. As $AB + BC > AC \geq 1$ we find that $S_1 \neq \{AB, BC\}$. Similarly, S_1 cannot contain two adjacent sides of the quadrilateral $ABCD$. Therefore, without loss of generality, we may assume that $S_1 = \{AD, BC\}$ and $S_2 = \{AB, CD\}$. Then $AD + BC = 1$ and $AC + BD = 2$.
We have $AD \cdot BC \leq \frac{1}{4} \cdot (AD + BC)^2 = \frac{1}{4}$ and $AB \cdot CD \leq \frac{1}{4} \cdot (AB + CD)^2 = 1$.
According to Ptolemy's inequality, we have

$$\frac{5}{4} = AC \cdot BD \leq AB \cdot CD + BC \cdot AD = 1 + \frac{1}{4} = \frac{5}{4},$$

hence we have equality all around, which means the quadrilateral $ABCD$ is cyclic, $AD = BC = \frac{1}{2}$ and $AB = CD = 1$, hence $ABCD$ is a rectangle of dimensions 1 and $\frac{1}{2}$.
There are many different ways of proving that this configuration is not possible. For example:
Suppose $ABCD$ is a rectangle with $AD = \frac{1}{2}$, $AB = 1$. Then we have $AC = BD = \frac{\sqrt{5}}{2}$ and $\Delta ANB \equiv \Delta CQD$ (ASA). Denoting $AM = x$, $MD = y$ we have $AN = 1 - x$, $BN = 1 - y$ and the following conditions need to be fulfilled for some $x, y \in [0; 1]$: $x^2 + y^2 = \frac{1}{4}$, $(1 - x)^2 + (1 - y)^2 = 1$ and, as $BD^2 = 1^2 + (2y - 1)^2$, we also need $1 + (2y - 1)^2 = \frac{5}{4}$. We obtain $y \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$. Similarly, $AC^2 = 1^2 + (2x - 1)^2$ yields $x \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$ but these values do not fulfill $x^2 + y^2 = \frac{1}{4}$, therefore such a configuration is not possible.