**Romanian proposal for JBMO 2013**

**1.** A rectangle in the plane is called *latticed* if its vertices have integer coordinates.
Prove that:
a) if a *latticed* rectangle has area 11, then its sides are parallel to the axes;
b) there exists a *latticed* rectangle of area 13 whose sides are not parallel to the axes.

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*Solution:* a) We can translate the rectangle such that one of the vertices becomes the origin. The coordinates of the vertices are $A(0,0)$, $B(a,b)$, $D(c,-\frac{ac}{b})$, $C(a+c,\, b-\frac{ac}{b})$. Suppose $b \neq 0$. Then the dimensions of the rectangle are $L = \sqrt{a^2+b^2}$ and $\ell = \frac{c\sqrt{a^2+b^2}}{b}$, hence the condition $L \cdot \ell = 11$ reduces to $c(a^2+b^2) = 11b$. We obtain that $c \neq 0$. From $11 \mid a^2+b^2$ it follows that $11 \mid a$, $11 \mid b$, which means $a = 11x$, $b = 11y$, $x, y \in \mathbb{N}$ with $c(x^2+y^2) = y$. We have $\mid c(x^2+y^2) \mid \geq x^2+y^2 \geq y^2 \geqslant \mid y \mid$ with equality if and only if $c = \pm 1$, $a = 0$, $b = \pm 11$, which means that the rectangle has its sides parallel to the axes.
b) Similarly, one arrives to $c(a^2+b^2) = 13b$, with the solution $a = 2$, $b = 3$, $c = 3$ which leads to a square whose sides are not parallel to the axes.