Algebra

A1. *Let a, b and c be positive real numbers such that $a+b+c=1$. Prove the inequality*

$$\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}+6 \geq 2\sqrt{2}\left(\sqrt{\frac{1-a}{a}}+\sqrt{\frac{1-b}{b}}+\sqrt{\frac{1-c}{c}}\right).$$

When does equality hold?

Solution. Replacing $1-a, 1-b, 1-c$ with $b+c, a+c, a+b$ respectively on the right hand side, the given inequality becomes

$$\frac{a+c}{b}+\frac{b+c}{a}+\frac{a+b}{c}+6 \geq 2\sqrt{2}\left(\sqrt{\frac{b+c}{a}}+\sqrt{\frac{a+c}{b}}+\sqrt{\frac{a+b}{c}}\right)$$

and equivalently

$$\left(\frac{a+c}{b}-2\sqrt{2}\sqrt{\frac{a+c}{b}}+2\right)+\left(\frac{b+c}{a}-2\sqrt{2}\sqrt{\frac{b+c}{a}}+2\right)+\left(\frac{a+b}{c}-2\sqrt{2}\sqrt{\frac{a+b}{c}}+2\right) \geq 0,$$

which can be written as

$$\left(\sqrt{\frac{a+c}{b}}-\sqrt{2}\right)^2+\left(\sqrt{\frac{b+c}{a}}-\sqrt{2}\right)^2+\left(\sqrt{\frac{a+b}{c}}-\sqrt{2}\right)^2 \geq 0,$$

which is true. The equality holds if and only if $(a+c)/b=(b+c)/a=(a+b)/c=2$, which together with the given condition $a+b+c=1$ immediately give $a=b=c=1/3$. ■

A2. *Let a, b, c be positive real numbers such that $abc=1$. Show that*

$$\frac{1}{a^3+bc}+\frac{1}{b^3+ca}+\frac{1}{c^3+ab} \leq \frac{(ab+bc+ca)^2}{6}.$$

Solution. By the AM-GM inequality we have $a^3+bc \geq 2\sqrt{a^3bc}=2\sqrt{a^2(abc)}=2a$ and so

$$\frac{1}{a^3+bc} \leq \frac{1}{2a}.$$

Similarly, $\frac{1}{b^3+ca} \leq \frac{1}{2b}, \frac{1}{c^3+ab} \leq \frac{1}{2c}$ and then

$$\frac{1}{a^3+bc}+\frac{1}{b^3+ca}+\frac{1}{c^3+ab} \leq \frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}=\frac{1}{2}\frac{ab+bc+ca}{abc} \leq \frac{(ab+bc+ca)^2}{6}.$$

Therefore, it is enough to prove $\frac{(ab+bc+ca)^2}{6} \leq \frac{(ab+bc+ca)^2}{6}$. This inequality is trivially shown to be equivalent to $3 \leq ab+bc+ca$ which is true because of the AM-GM inequality: $3 = \sqrt[3]{(abc)^2} \leq ab+bc+ca$. ■

A3. *Let a, b, c be positive real numbers such that $a+b+c = a^2+b^2+c^2$. Show that*

$$\frac{a^2}{a^2+ab} + \frac{b^2}{b^2+bc} + \frac{c^2}{c^2+ca} \geq \frac{a+b+c}{2}.$$

Solution. By the Cauchy-Schwarz inequality it is

$$\left(\frac{a^2}{a^2+ab} + \frac{b^2}{b^2+bc} + \frac{c^2}{c^2+ca}\right)\left((a^2+ab)+(b^2+bc)+(c^2+ca)\right) \geq (a+b+c)^2$$

$$\Rightarrow \quad \frac{a^2}{a^2+ab} + \frac{b^2}{b^2+bc} + \frac{c^2}{c^2+ca} \geq \frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca}.$$

So it is enough to prove $\frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca} \geq \frac{a+b+c}{2}$, that is to prove

$$2(a+b+c) \geq a^2+b^2+c^2+ab+bc+ca.$$

Substituting $a^2+b^2+c^2$ for $a+b+c$ into the left hand side we wish equivalently to prove

$$a^2+b^2+c^2 \geq ab+bc+ca.$$

But the $a^2+b^2 \geq 2ab$, $b^2+c^2 \geq 2bc$, $c^2+a^2 \geq 2ca$ which by addition imply the desired inequality. ■

A4. *Solve the following equation for $x, y, z \in \mathbb{N}$*

$$\left(1+\frac{x}{y+z}\right)^2 + \left(1+\frac{y}{z+x}\right)^2 + \left(1+\frac{z}{x+y}\right)^2 = \frac{27}{4}$$

Solution 1. Call $a = 1+\frac{x}{y+z}$, $b = 1+\frac{y}{z+x}$, $c = 1+\frac{z}{x+y}$ to get

$$a^2+b^2+c^2 = \frac{27}{4}.$$

Since it is also true that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 2,$$

the quadratic-harmonic means inequality implies

$$\frac{3}{2} = \sqrt{\frac{a^2+b^2+c^2}{3}} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = \frac{3}{2}$$

So the inequality in the middle holds as an equality, and this happens whenever $a = b = c$, from which $1 + \dfrac{x}{y+z} = 1 + \dfrac{y}{z+x} = 1 + \dfrac{z}{x+y}$.

But $1+\dfrac{x}{y+z} = 1+\dfrac{y}{z+x} \Leftrightarrow x^2+xz = y^2+yz \Leftrightarrow (x-y)(x+y) = z(y-x)$ and the two sides of this equality will be of different sign, unless $x = y$ in which case both sides become 0. So $x = y$, and similarly $y = z$, thus $x = y = z$.

Indeed, any triad of equal natural numbers $x = y = z$ is a solution for the given equation, and so these are all its solutions. ∎

Solution 2. The given equation is equivalent to

$$\frac{27}{4} = (x+y+z)^2\left(\frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} + \frac{1}{(x+y)^2}\right).$$

Now observe that by the well known inequality $a^2+b^2+c^2 \geq ab+bc+ca$, with $\dfrac{1}{y+z}, \dfrac{1}{z+x}, \dfrac{1}{x+y}$ in place of a, b, c, we get:

$$\begin{aligned}
\frac{27}{4} &= (x+y+z)^2\left(\frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} + \frac{1}{(x+y)^2}\right) \\
&\geq (x+y+z)^2\left(\frac{1}{(y+z)(z+x)} + \frac{1}{(z+x)(x+y)} + \frac{1}{(x+y)(y+z)}\right) = \frac{2(x+y+z)^3}{(x+y)(y+z)(z+x)} \\
&= \frac{(2(x+y+z))^3}{4(x+y)(y+z)(z+x)} = \frac{((x+y)+(y+z)+(z+x))^3}{4(x+y)(y+z)(z+x)} \overset{\text{AM-GM}}{\geq} \frac{\left(3\sqrt[3]{(x+y)(y+z)(z+x)}\right)^3}{4(x+y)(y+z)(z+x)} \\
&= \frac{27}{4}.
\end{aligned}$$

This means all inequalities in the above calculations are equalities, and this holds exactly whenever $x+y = y+z = z+x$, that is $x = y = z$. By the statement's demand we need to have a, b, c integers. And conversely, any triad of equal natural numbers $x = y = z$ is indeed a solution for the given equation, and so these are all its solutions. ∎

A5. *Find the largest positive integer n for which the inequality*

$$\frac{a+b+c}{abc+1} + \sqrt[n]{abc} \leq \frac{5}{2} \qquad (1)$$

holds for all $a, b, c \in [0,1]$. Here $\sqrt[1]{abc} = abc$.

Solution. Let n_{max} be the sought largest value of n, and let $E_{a,b,c}(n) = \dfrac{a+b+c}{abc+1} + \sqrt[n]{abc}$. Then $E_{a,b,c}(m) - E_{a,b,c}(n) = \sqrt[m]{abc} - \sqrt[n]{abc}$ and since $abc \leq 1$ we clearly have $E_{a,b,c}(m) \geq E_{a,b,c}(n)$ for $m \geq n$. So if $E_{a,b,c}(n) \geq \frac{5}{2}$ for some choice of $a, b, c \in [0,1]$, it must be $n_{max} \leq n$. We use this remark to determine the upper bound $n_{max} \leq 3$ by plugging some particular values of a, b, c into the given inequality as follows:

For $(a, b, c) = (1, 1, c), c \in [0, 1]$, inequality (1) implies $\dfrac{c+2}{c+1} + \sqrt[n]{c} \leq \dfrac{5}{2} \Leftrightarrow \dfrac{1}{c+1} + \sqrt[n]{c} \leq$

$\frac{3}{2}$. Obviously, every $x \in [0;1]$ is written as $\sqrt[n]{c}$ for some $c \in [0;1]$. So the last inequality is equivalent to:

$$\frac{1}{x^n+1} + x \le \frac{3}{2} \Leftrightarrow 2 + 2x^{n+1} + 2x \le 3x^n + 3 \Leftrightarrow 3x^n + 1 \ge 2x^{n+1} + 2x$$
$$\Leftrightarrow \quad 2x^n(1-x) + (1-x) + (x-1)(x^{n-1} + \cdots + x) \ge 0$$
$$\Leftrightarrow \quad (1-x)[2x^n + 1 - (x^{n-1} + x^{n-2} + \ldots + x)] \ge 0, \ \forall x \in [0,1].$$

For $n = 4$, the left hand side of the above becomes $(1-x)(2x^4 + 1 - x^3 - x^2 - x) = (1-x)(x-1)(2x^3 + x^2 - 1) = -(1-x)^2(2x^3 + x^2 - 1)$ which for $x = 0.9$ is negative. Thus $n_{max} \le 3$ as claimed.

Now, we shall prove that for $n = 3$ inequality (1) holds for all $a, b, c \in [0,1]$, and this would mean $n_{max} = 3$. We shall use the following Lemma:

Lemma. For all $a, b, c \in [0;1] : a + b + c \le abc + 2$.

Proof of the Lemma: The required result comes by adding the following two inequalities side by side

$$0 \le (a-1)(b-1) \Leftrightarrow a + b \le ab + 1 \quad \Leftrightarrow \quad a + b - ab \le 1$$
$$0 \le (ab-1)(c-1) \quad \Leftrightarrow \quad ab + c \le abc + 1.$$

Because of the Lemma, our inequality (1) for $n = 3$ will be proved if the following weaker inequality is proved for all $a, b, c \in [0,1]$:

$$\frac{abc+2}{abc+1} + \sqrt[3]{abc} \le \frac{5}{2} \Leftrightarrow \frac{1}{abc+1} + \sqrt[3]{abc} \le \frac{3}{2}.$$

Denoting $\sqrt[3]{abc} = y \in [0;1]$, this inequality becomes:

$$\frac{1}{y^3+1} + y \le \frac{3}{2} \Leftrightarrow 2 + 2y^4 + 2y \le 3y^3 + 3 \Leftrightarrow -2y^4 + 3y^3 - 2y + 1 \ge 0$$
$$\Leftrightarrow \quad 2y^3(1-y) + (y-1)y(y+1) + (1-y) \ge 0 \Leftrightarrow (1-y)(2y^3 + 1 - y^2 - y) \ge 0.$$

The last inequality is obvious because $1-y \ge 0$ and $2y^3+1-y^2-y = y^3+(y-1)^2(y+1) \ge 0$.

■

Geometry

G1. *Let ABC be an equilateral triangle, and P a point on the circumcircle of the triangle ABC and distinct from A, B and C. If the lines through P and parallel to BC, CA, AB intersect the lines CA, AB, BC at M, N and Q respectively, prove that M, N and Q are collinear.*

Solution. Without any loss of generality, let P be in the minor arc of the chord AC as in Figure 1. Since $\angle PNA = \angle NPM = 60°$ and $\angle NAM = \angle PMA = 120°$, it follows that the points A, M, P and N are concyclic. This yields

$$\angle NMP = \angle NAP. \tag{2}$$

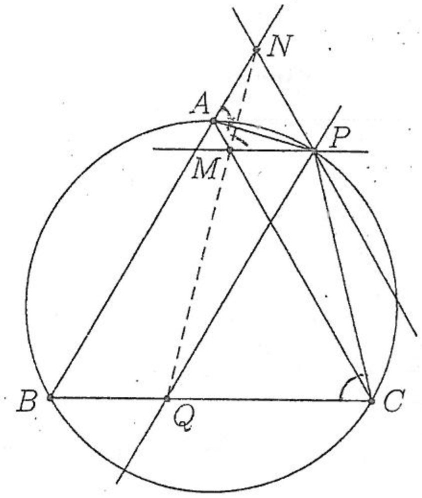

Figure 1: Exercise G1.

Similarly, since $\angle PMC = \angle MCQ = 60°$ and $\angle CQP = 60°$, it follows that the points P, M, Q and C are concyclic. Thus

$$\angle PMQ = 180° - \angle PCQ = 180° - \angle NAP \overset{(2)}{=} 180° - \angle NMP.$$

This implies $\angle PMQ + \angle NMP = 180°$, which shows that M, N and Q belong to the same line. ■

G2. *Let ABC be an isosceles triangle with $AB = AC$. Let also $c(K, KC)$ be a circle tangent to the line AC at point C which it intersects the segment BC again at an interior point H. Prove that $HK \perp AB$.*

Solution 1. Let lines KH, AB intersect at M (Figure 5a). From the quadrelateral $KMAC$ we have

$\angle KMA = 360^o - \angle A - \angle ACK - \angle CKM = 360^o - \angle A - 90^o - (180^0 - 2\angle KCH) = 90 - \angle A + 2\angle KCH = 90 - \angle A + 2(90^0 - \angle ACB) = 270^0 - \angle A - 2\angle ACB = 270 - \angle A - \angle ACB - \angle ABC = 270^o - 180^0 = 90^0$.

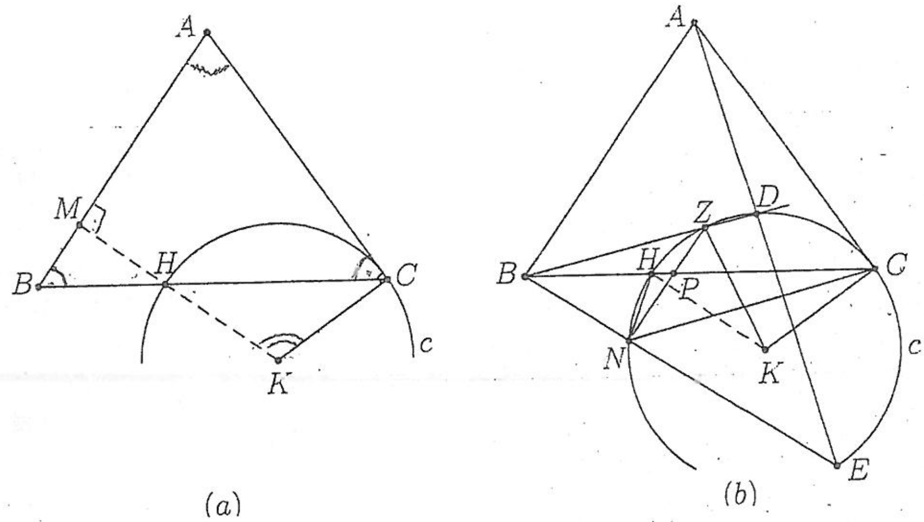

Figure 2: Exercise G2.

so $KH \perp AB$ as wanted. ■

Solution 2. Let D be a point on c such that $AD < AC$, and let E, Z be the second points of intersection of lines AD and BD with c respectively. Let also N be the second point of intersection of line BE with the circle c. Figure 5b shows Z between B, D. The argument below can be trivially modified to apply in case D is in the segment B, Z as well. It is

$$AB^2 = AC^2 = AD \cdot AE \Rightarrow \frac{AB}{AE} = \frac{AD}{AB}.$$

This relation and the fact that $\angle BAE = \angle BAD$ implies that the triangles ABE, ADB are similar. Thus $\angle ABE = \angle ADB$. Also from the cyclic quadrilateral we get $\angle ADB = \angle ZNE$. Therefore $\angle ABE = \angle ZNE$, so $AB || NZ$.

Call P the intersection point of BC, NZ. Since AC is tangent to c it is

$$\angle CEH = \angle BCA \tag{3}$$

and then

$$\begin{aligned} & \angle ZNH + \angle CNZ = \angle HNC = \angle CEH \stackrel{(3)}{=} \angle BCA = \angle ABC = \angle ZPC = \angle BCN + \angle CNZ \\ \Rightarrow \quad & \angle ZNH = \angle BCN \\ \Rightarrow \quad & \angle ZNH = \angle HZN \end{aligned}$$

Therefore H is the midpoint of the arc NZ, so $KH \perp NZ$ and as $AB || NZ$ we finally get $KH \perp AB$ as wanted. ■

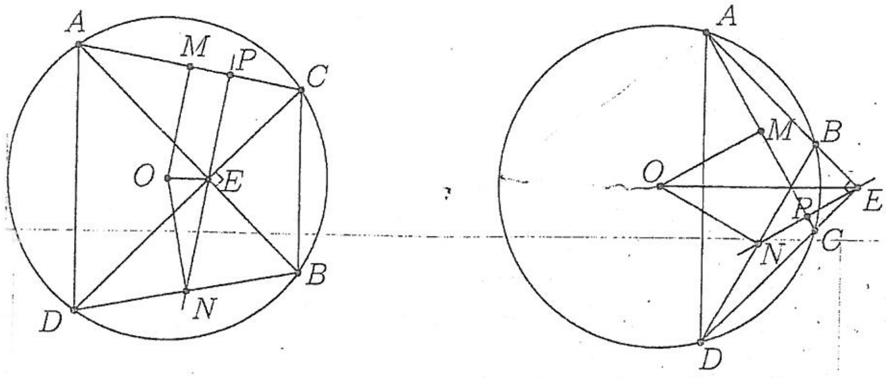

Figure 3: Exercise G3.

G3. *Let AB and CD be chords in a circle of center O with A, B, C, D distinct, and let the lines AB and CD meet at a right angle at point E. Let also M and N be the midpoints of AC and BD respectively. If $MN \perp OE$, prove that $AD||BC$.*

Solution. E can be inside, or outside the circle (Figure 3) but the proof below holds in both cases; notice that E cannot be on the circle as A, B, C, D are distinct. Let lines AC and NE meet at point P. Then $EN = DN = BN$ (median in a right triangle), so $\angle PEC = \angle NED = \angle NDE = \angle BDC = \angle BAC = \angle EAP$. Now $AB \perp CD$ so $EN \perp AC$. But $OM \perp AC$ so $OM||EN$. Similarly $ON||EM$ so $NEMO$ is a parallelogram (possibly degenerated). As $MN \perp OE$, this parallelogram is a rhombus. Then the chords AC and BD, being equidistant from O, are equal. Hence their minor arcs are equal, which means that either $AD||BC$ or $AB||CD$; the latter contradicts the fact that AB and CD meet at E. ■

G4. *Let ABC be an acute-angled triangle with circumcircle Γ, and let O, H be the triangle's circumcenter and orthocenter respectively. Let also A' be the point where the angle bisector of angle BAC meets Γ. If $A'H = AH$, find the measure of angle BAC.*

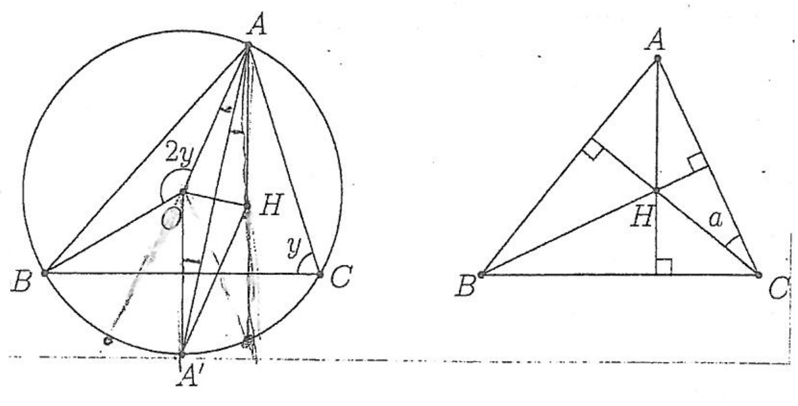

Figure 4: Exercise G4.

Solution. The segment AA' bisects $\angle OAH$: if $\angle BCA = y$ (Figure 4), then $\angle BOA = 2y$, and since $OA = OB$, it is $\angle OAB = \angle OBA = 90^o - y$. Also since $AH \perp BC$, it is

$\angle HAC = 90^o - y = \angle OAB$ and the claim follows.

Since AA' bisects $\angle OAH$ and $A'H = AH. OA' = OA$, we have that the isosceles triangles OAA', HAA' are equal. Thus

$$AH = OA = R \tag{4}$$

where R is the circumradius of triangle ABC.

Call $\angle ACH = a$ and recall by the law of sines that $AH = 2R' \sin a$, where R' is the circumradius of triangle AHC. Then (4) implies

$$R = 2R' \sin a \tag{5}$$

But notice that $R = R'$ because $\dfrac{AC}{\sin(AHC)} = 2R', \dfrac{AC}{\sin(ABC)} = 2R$ and $\sin(AHC) = \sin(180^o - ABC) = \sin(ABC)$. So (5) gives $1 = 2 \sin a$, and a as an acute angle can only be 30^o. Finally, $\angle BAC = 90^o - a = 60^o$. ■

Remark. The steps in the above proof can be traced backwards making the converse also true, that is: If $\angle BAC = 60^0$ then $A'H = AH$.

2-3 **G5.** *Let the circles k_1 and k_2 intersect at two distinct points A and B, and let t be a common tangent of k_1 and k_2 that touches them at M and N respectively. If $t \perp AM$ and $MN = 2AM$, evaluate $\angle NMB$.*

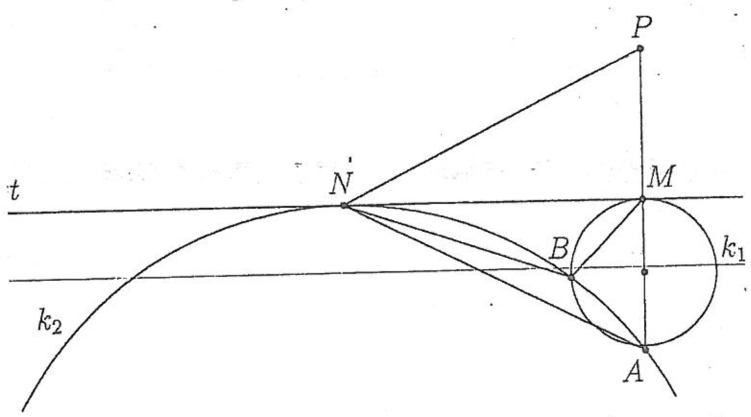

Figure 5: Exercise G5.

Solution. Let P be the symmetric of A with respect to M (Figure 5). Then $AM = MP$ and $t \perp AP$, hence the triangle APN is isosceles with AP as its base, so $\angle NAP = \angle NPA$. We have $\angle BAP = \angle BAM = \angle BMN$ and $\angle BAN = \angle BNM$. Thus

$$180^o - \angle NBM = \angle BNM + BMN = \angle BAN + \angle BAP = \angle NAP = \angle NPA,$$

so the quadrangle $MBNP$ is cyclic (since the points B and P lie on different sides of MN). Hence $\angle APB = \angle MPB = \angle MNB$ and the triangles APB and MNB are congruent ($MN = 2AM = AM + MP = AP$). From that we get $AB = MB$, i.e. the triangle AMB is isosceles, and since t is tangent to k_1 and perpendicular to AM, the center of k_1 is on AM, hence AMB is a right-angled triangle. From the last two statements we infer $\angle AMB = 45^0$, and so $\angle NMB = 90^o - \angle AMB = 45^o$. ∎

G6. *Let O_1 be a point in the exterior of the circle $c(O, R)$ and let O_1N, O_1D be the tangent segments from O_1 to the circle. On the segment O_1N consider the point B such that $BN = R$. Let the line from B parallel to ON intersect the segment O_1D at C. If A is a point on the segment O_1D other than C so that $BC = BA = a$, and if $c'(K, r)$ is the incircle of the triangle O_1AB; find the area of ABC in terms of a, R, r.*

Solution. Obviously, the segment BC is tangent to the circle c. Let M be the point of tangency (Figure 6). Call Q, M the tangency points of BA, BC with c' and c respectively, and call H the midpoint of segment AC. It is well known that

$$AQ = \frac{1}{2}(AO_1 + AB - BO_1) \text{ and } CM = \frac{1}{2}(BO_1 + BC - CO_1).$$

and so

$$AQ + CM = \frac{1}{2}(2BC - AC) = a - \frac{1}{2}AC = a - HC.$$

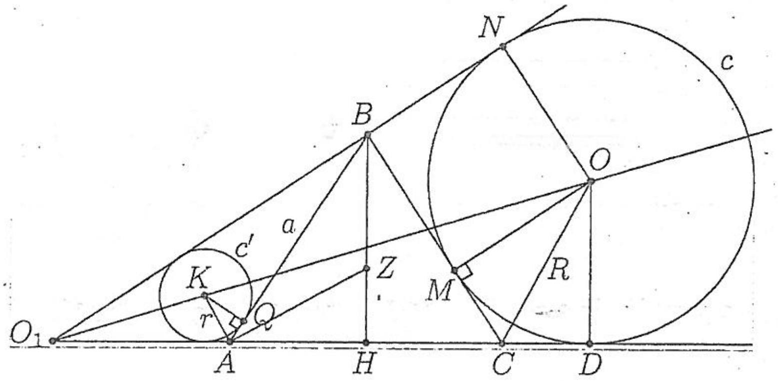

Figure 6: Exercise G6.

The triangles KAQ and OCM are similar and this implies

$$\frac{KQ}{OM} = \frac{AQ}{CM} \Leftrightarrow \frac{KQ}{AQ} = \frac{OM}{CM} \Leftrightarrow \frac{r}{AQ} = \frac{R}{CM} = \frac{R+r}{AQ+CM} = \frac{R+r}{a-HC} \Leftrightarrow$$

$$\frac{r}{AQ} = \frac{R+r}{a-HA}. \tag{6}$$

If AZ is the bisector segment of triangle BAH it holds

$$\angle AZH = 90^o = \frac{1}{2}\angle BAC \text{ and } \angle KAQ = \frac{1}{2}(180^o - \angle BAC) = 90^o - \frac{1}{2}\angle BAC.$$

Therefore, from the similar triangles KQA and AHZ we get

$$\frac{AH}{ZH} = \frac{r}{AQ} \overset{(6)}{=} \frac{R+r}{a-HA}. \tag{7}$$

Also, from the bisector-theorem in triangle ABH it holds

$$ZH = \frac{AH \cdot BH}{a + AH}$$

and from (7) it follows

$$\frac{R+r}{a-HA} = \frac{AH}{\frac{AH \cdot BH}{a+AH}} \Rightarrow R + r = \frac{a^2 - AH^2}{BH} = \frac{BH^2}{BH} = BH.$$

So

$$HA^2 = a^2 - (R+r)^2 \Leftrightarrow HA = \sqrt{a^2 - (R+r)^2}$$

and finally the area of triangle ABC in terms of a, R, r is:

$$(ABC) = AH \cdot BH = (R+r)\sqrt{a^2 - (R+r)^2}$$

■

G7. *Let $MNPQ$ be a square of side length 1, and A, B, C, D points on the sides MN, NP, PQ, and QM respectively such that $AC \cdot BD = \frac{5}{4}$. Can the set $\{AB, BC, CD, DA\}$ be partitioned into two subsets S_1 and S_2 of two elements each, so that each one of the sums of the elements of S_1 and S_2 are positive integers?*

Solution. The answer is negative.

Suppose such a partitioning was possible (Figure 7). Then $AB + BC + CD + DA \in \mathbb{N}$. But $(AB + BC) + (CD + DA) > AC + AC \geq 2$, hence $AB + BC + CD + DA > 2$.

On the other hand, $AB + BC + CD + DA < (AN + NB) + (BP + PC) + (CQ + QD) + (DM + MA) = 4$, hence $AB + BC + CD + DA = 3$.

Obviously one of the sums of the elements of S_1 and S_2 must be 1 and the other 2. Without any loss of generality, we may assume that the sum of the elements of S_1 is 1 and the sum of the elements of S_2 is 2. As $AB + BC > AC \geq 1$ we find that $S_1 \neq \{AB, BC\}$. Similarly, S_1 cannot contain two adjacent sides of the quadrilateral $ABCD$. Therefore, without any loss of generality, we may assume that $S_1 = \{AD, BC\}$ and $S_2 = \{AB, CD\}$. Then $AD + BC = 1$ and $AB + CD = 2$.

We have $AD \cdot BC \leq \frac{1}{4} \cdot (AD + CB)^2 = \frac{1}{4}$ and $AB \cdot CD \leq \frac{1}{4} \cdot (AB + CD)^2 = 1$.

According to Ptolemy's inequality, we have

$$\frac{5}{4} = AC \cdot BD \leq AB \cdot CD + AD \cdot BC = \frac{1}{4} + 1 = \frac{5}{4},$$

hence we have equality all around, which means the quadrilateral $ABCD$ is cyclic, $AD = BC = \frac{1}{2}$ and $AB = CD = 1$, hence $ABCD$ is a rectangle of dimensions 1 and $\frac{1}{2}$.

There are many different ways of proving that this configuration is not possible. For example:

Suppose $ABCD$ is a rectangle with $AD = \frac{1}{2}$, $AB = 1$. Then we have $AC = BD = \frac{\sqrt{5}}{2}$ and $\Delta ANB \equiv \Delta CQD$ (Angle-Side-Angle). Denoting $AM = x$, $MD = y$ we have $AN =$

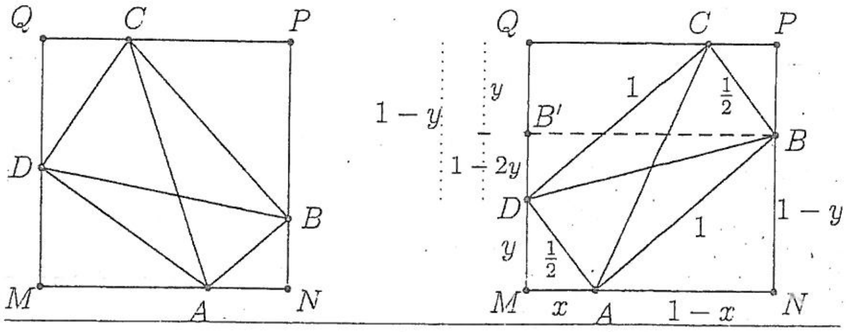

Figure 7: Exercise G7.

$1-x$, $BN = 1-y$ and the following conditions need to be fulfilled for some $x, y \in [0;1]$ (Pythagorean Theorem in triangles $AMD, ANB, BB'C$, where B' is the projection of B on MQ):

$$x^2 + y^2 = \frac{1}{4}, \quad (1-x)^2 + (1-y)^2 = 1 \text{ and } 1 + (2y-1)^2 = \frac{5}{4}. \tag{8}$$

But $1+(2y-1)^2 = \frac{5}{4}$ implies $y \in \left\{\frac{1}{4}, \frac{3}{4}\right\}$. If $y = \frac{3}{4}$, then $x^2+y^2 = \frac{1}{4}$ cannot hold. If on the other hand $y = \frac{1}{4}$, then $(1-x)^2+(1-y)^2 = 1$ implies $x = 0$, but then $(1-x)^2+(1-y)^2 = 1$ cannot hold. Therefore such a configuration is not possible. ∎

Combinatorics

C1. *Along a round table are arranged 11 cards with the names (all distinct) of the 11 members of the 16^{th} JBMO Problem Selection Committee. The distances between each two consecutive cards are equal. Assume that in the first meeting of the Committee none of its 11 members sits in front of the card with his name. Is it possible to rotate the table by some angle so that at the end at least two members of sit in front of the card with their names?*

Solution. Yes it is: Rotating the table by the angles $\frac{360^\circ}{11}$, $2\cdot\frac{360^\circ}{11}$, $3\cdot\frac{360^\circ}{11}$, ..., $10\cdot\frac{360^\circ}{11}$, we obtain 10 new positions of the table. By the assumption, it is obvious that every one of the 11 members of the Committee will be seated in front of the card with his name in exactly one of these 10 positions. Then by the Pigeonhole Principle there should exist one among these 10 positions in which at least two of the $11(>10)$ members of the Committee will be placed in their positions, as claimed. ■

C2. *n nails nailed on a board are connected by two via a string. Each string is colored in one of n given colors. For any three colors there exist three nails connected by two with strings in these three colors. Can n be: (a) 6, (b) 7?*

Solution. (a) The answer is no:
Suppose it is possible. Consider some color, say blue. Each blue string is the side of 4 triangles formed with vertices on the given points. As there exist $\binom{5}{2}=\frac{5\cdot4}{2}=10$ pairs of colors other than blue, and for any such pair of colors together with the blue color there exists a triangle with strings in these colors, we conclude that there exist at least 3 blue strings (otherwise the number of triangles with a blue string as a side would be at most $2\cdot4=8$, a contradiction). The same is true for any color, so altogether there exist at least $6\cdot3=18$ strings, while we have just $\binom{6}{2}=\frac{6\cdot5}{2}=15$ of them.

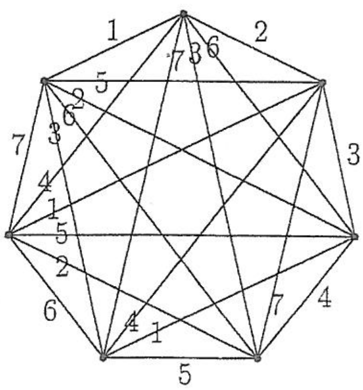

Figure 8: Exercise C2.

(b) The answer is yes (Figure 8):

Put the nails at the vertices of a regular 7-gon (Figure 8) and color each one of its sides in a different color. Now color each diagonal in the color of the unique side parallel to it. It can be checked directly that each triple of colors appears in some triangle (because of symmetry, it is enough to check only the triples containing the first color). ■

Remark. The argument in (a) can be applied to any even n. The argument in (b) can be applied to any odd $n = 2k + 1$ as follows: first number the nails as $0, 1, 2 \ldots, 2k$ and similarly number the colors as $0, 1, 2 \ldots, 2k$. Then connect nail x with nail y by a string of color $x + y(\text{mod} n)$. For each triple of colors (p, q, r) there are vertices x, y, z connected by these three colors. Indeed, we need to solve $(\text{mod} n)$ the system

$$(*)(x + y \equiv p,\ x + z \equiv q,\ y + z \equiv r)$$

Adding all three, we get $2(x+y+z) \equiv p+q+r$ and multiplying by $k+1$ we get $x+y+z \equiv (k+1)(p+q+r)$. We can now find x, y, z from the identities $(*)$.

3 **C3.** *In a circle of diameter* 1 *consider* 65 *points no three of which are collinear. Prove that there exist* 3 *among these points which form a triangle with area less then or equal to* $\frac{1}{72}$.

Solution. Lemma: If a triangle ABC lies in a rectangle $KLMN$ with sides $KL = a$ and $LM = b$, then the area of the triangle is less then or equal to $\frac{ab}{2}$.

Proof of the lemma: Without any loss of generality assume that among the distance of A, B, C from KL, that of A is between the other two. Let ℓ be the line through A and parallel to KL. Let D be the intersection of ℓ, BC and x, y the distances of B, C from ℓ respectively. Then the area of ABC equals $\frac{AD(x+y)}{2} \leq \frac{ab}{2}$, since $AD \leq a$ and $x + y \leq b$ and we are done.

Now back to our problem, let us cover the circle with 24 squares of side $\frac{1}{6}$ and 8 other irregular and equal figures as shown in Figure 9, with boundary consisting of an arc on the circle and three line segments. Call $S = ADNM$ one of these figures. One of the line segments in the boundary of S is of length $AD = AB - DB = \sqrt{AC^2 - BC^2} - \frac{2}{6} = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{6}\right)^2} - \frac{1}{3} = \frac{\sqrt{2}-1}{3}$. The boundary segment MN goes through the center C of the circle, forming with the horizontal lines an angle of 45^0. The point in S with maximum distance from the boundary segment AB is the endpoint M of the arc on the boundary of S. This distance equals $ME = MF - EF \overset{CMF=\text{isosceles}}{=} \frac{\sqrt{2}}{2}CM - \frac{1}{6} = \frac{\sqrt{2}}{4} - \frac{1}{6} = \frac{3\sqrt{2}-2}{12}$.

So S can be put inside a rectangle R with sides parallel to AD, ND of lengths $\frac{\sqrt{2}-1}{3}$ and $\frac{3\sqrt{2}-2}{12}$. So the triangle formed by any three points inside this figure, has an area less or equal to $\frac{1}{2} \cdot \frac{\sqrt{2}-1}{3} \cdot \frac{3\sqrt{2}-2}{12} = \frac{8-5\sqrt{2}}{72} < \frac{1}{72}$.

Also, the triangle formed by any three points inside any square of side $\frac{1}{6}$, has an area less or equal to $\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{72}$.

By the Pigeonhole Principle, we know that among the 65 given points there exist 3 inside the same one of the 32 squares and irregular figures of the picture covering the given circle. Then according to the above, the triangle formed by these 3 points has an area not exceeding $\frac{1}{72}$ as wanted. ■

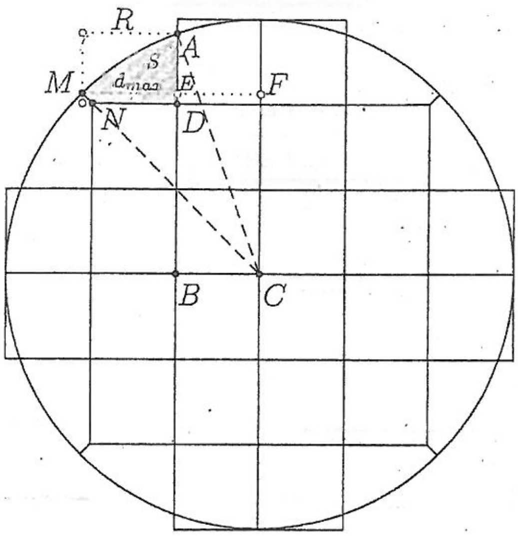

Figure 9: Exercise C3.

Number Theory

NT1. *If a, b are integers and $s = a^3 + b^3 - 60ab(a+b) \geq 2012$, find the least possible value of s.*

Solution. It is $s = (a+b)^3 - 63ab(a+b)$ which gives the same residue module 7 as $(a+b)^3$. But the residues modulo 7 of perfect cubes can only be $0, 1$ or 6. So the residue of s modulo 7 is $0, 1$ or 6. Now for $a = 6, b = -1$ we get $s = 2015 \geq 2012$ and this is the least possible value of s because the numbers $2012, 2013, 2014$ give $3, 4$ and 5 as residues mod7 which are distinct from $0, 1, 6$, and so $2012, 2013, 2014$ cannot be s for any choice of a, b. ∎

NT2. *Do there exist prime numbers p and q such that $p^2(p^3 - 1) = q(q+1)$?*

Solution. Write the given equation in the form

$$p^2(p-1)(p^2+p+1) = q(q+1). \tag{9}$$

First observe that it must not be $p = q$, since in this case the left hand side of (9) is greater than its right hand side. Hence, since p and q are distinct primes, (9) immediately yields $p^2 \mid q+1$, that is

$$q = ap^2 - 1 \tag{10}$$

for some $a \in \mathbb{N}$. Since p and q are both primes, by (9) we get the following cases:

Case 1: $q \mid p - 1$, that is

$$p = bq + 1 \tag{11}$$

for some $b \in \mathbb{N}$. Substituting (11) into (10), and using the fact that $a \geq 1$ and $b \geq 1$, we obtain

$$q = a(bq+1)^2 - 1 \geq (q+1)^2 - 1 = q^2 + 2q,$$

a contradiction.

Case 2: $q \mid p^2 + p + 1$, that is

$$p^2 + p + 1 = bq \tag{12}$$

for some $b \in \mathbb{N}$. Substituting (10) into (12), we get

$$p^2 + p + 1 = b(ap^2 - 1) \tag{13}$$

If $a \geq 2$, then from (13) it follows that

$$p^2 + p + 1 \geq 2p^2 - 1,$$

or equivalently, $p + 1 \geq (p-1)(p+1)$, that is, $(p+1)(2-p) \geq 0$. This implies that $p = 2$, and so $q \mid 2^2 + 2 + 1 = 7$. Hence, $q = 7$, but the pair $p = 2$ and $q = 7$ does not satisfy the equation (9).

Hence, it must be $a = 1$. Then if $b \geq 3$, (13) implies

$$p^2 + p + 1 \geq 3(p^2 - 1),$$

or equivalently, $4 \geq p(2p - 1)$, which is obviously impossible.

Thus, it must be $a = 1$ and $b \in \{1, 2\}$. For $a = b = 1$, (13) implies that $p = 2$, which by (12) again yields $q = 7$, which is impossible. Finally, for $a = 1$ and $b = 2$, (13) gives $p(p-1) = 3$, which is clearly not satisfied for any prime p.

Hence, there do not exist prime numbers p and q which satisfy given equation. ∎

NT3. *Decipher the equality*

$$(\overline{VER} - \overline{IA}) : (\overline{GRE} + \overline{ECE}) = G^{R^E},$$

assuming that the number $\overline{GREECE}$ has a maximum value. It is supposed that each letter corresponds to a unique digit from 0 to 9 and different letters correspond to different digits, and also that all letters G, E, V and I are different from 0. Also, the notation $\overline{a_n \ldots a_1 a_0}$ stands for the number $a_n \cdot 10^n + \cdots + 10^1 \cdot a_1 + a_0$.

Solution. Denote

$$x = \overline{VER} - \overline{IA}, \; y = \overline{GRE} + \overline{ECE}, \; z = G^{R^E}.$$

Then obviously, we have

$$\begin{aligned} &(201 + 131 \text{ or } 231 + 101) \leq y \leq (879 + 969 \text{ or } 869 + 979 \text{ or } 769 + 989) \\ \Rightarrow \; &332 \leq y \leq 1848 \Rightarrow 102 - 98 \leq x \leq 987 - 10 \Rightarrow 4 \leq x \leq 977, \end{aligned}$$

hence it follows that

$$\frac{4}{1848} \leq \frac{x}{y} = z \leq \frac{977}{332} \Rightarrow 1 \leq z \leq 2.$$

This shows that $z = G^{R^E} \in \{1, 2\}$. Hence, if $R \geq 1$, then $R^E \geq 1$, which implies that $2 \geq G^{R^E} \geq G$. Thus, if $R \geq 1$, then it must be $G \leq 2$. In view of this and the assumption of the problem that the number $\overline{GREECE}$ has a maximum value, we will consider the case when $R = 0$ hoping to get a solution with $G > 2$. Then $G^{R^E} = G^0 = 1$ for all digits G and E with $1 \leq G, E \leq 9$, and therefore, the above equality becomes

$$\overline{VER} - \overline{IA} = \overline{GRE} + \overline{ECE},$$

which substituting $R = 0$, can be written as

$$\overline{VE0} = \overline{G0E} + \overline{ECE} + \overline{IA}. \tag{14}$$

Now we consider the following cases:

Case 1: $G = 9$. Then $V \leq 8$, so $\overline{VE0} \leq 900$, while the right hand side of (1) is greater than 900. This is impossible, and no solution exists in this case.

Case 2: $G = 8$. Then (14) becomes

$$\overline{VE0} = \overline{80E} + \overline{ECE} + \overline{IA}, \tag{15}$$

hence it immediately follows that $V = 9$. For $V = 9$, (15) becomes

$$\overline{9E0} = \overline{80E} + \overline{ECE} + \overline{IA}, \tag{16}$$

Notice that for $E \geq 2$, the right hand side of (16) is greater than 1000, while the left hand side of (16) is less than 1000. Therefore, it must be $E \leq 1$, that is, $E = 1$ in view of the fact that $R = 0$. Substituting $E = 1$ into (16), we get

$$\overline{910} = \overline{801} + \overline{1C1} + \overline{IA}, \tag{17}$$

hence it follows that

$$109 = \overline{1C1} + \overline{IA}. \tag{18}$$

But the right hand side of (18) is greater than 121. This shows that $G = 8$ does not lead to any solution.

Case 3: $G = 7$. Then (14) becomes

$$\overline{VE0} = \overline{70E} + \overline{ECE} + \overline{IA}. \tag{19}$$

Thus it must be $V \geq 8$.

Subcase 3(a): $V = 8$. Then (19) gives

$$\overline{8E0} = \overline{70E} + \overline{ECE} + \overline{IA}, \tag{20}$$

hence we immediately obtain $E = 1$ (since the right hand side of (6) must be lass than 900). For $E = 1$, (20) reduces to

$$109 = \overline{1C1} + \overline{IA}, \tag{21}$$

which is impossible since $\overline{1C1} \geq 121$.

Subcase 3(b): $V = 9$. Then (19) gives

$$\overline{9E0} = \overline{70E} + \overline{ECE} + \overline{IA}, \tag{22}$$

hence we immediately obtain $E \leq 2$ (since the right hand side of (7) must be less than 1000). For $E = 2$, (22) reduces to

$$218 = \overline{2C2} + \overline{IA}, \tag{23}$$

which is impossible since $\overline{2C2} \geq 232$. Finally, for $E = 1$, (22) reduces to

$$209 = \overline{1C1} + \overline{IA}. \tag{24}$$

Since it is required that the number $\overline{GREECE}$ has a maximum value, taking $C = 8$ into (24) we find that

$$28 = \overline{IA}, \tag{25}$$

which yields $8 = A = C$. This is impossible since must be $A \neq C$. Since $C \neq G = 7$, then taking $C = 6$ into (24) we obtain

$$48 = \overline{IA}, \tag{26}$$

hence we have $I = 4$ and $A = 8$. Previously, we have obtain $G = 7$, $R = 0$, $V = 9$, $E = 1$ and $C = 6$. For these values, we obtain that $\overline{GREECE} = 701161$ is the desired maximum value. ■

NT4. Determine all triples (m, n, p) satisfying

$$n^{2p} = m^2 + n^2 + p + 1 \tag{27}$$

where m and n are integers and p is a prime number.

Solution. By Fermat's theorem $n^{2p} \equiv n^2 (\text{mod} p)$, therefore $m^2+n^2+p+1 \equiv n^2 (\text{mod} p) \Rightarrow m^2 \equiv -1 (\text{mod} p)$.

Case 1: $p = 4k+3$. We have $(m^2)^{2k+1} \equiv (-1)^{2k+1} (\text{mod} p)$. Therefore,

$$m^{p-1} \equiv -1 (\text{mod} p) \tag{28}$$

and p does not divide m. On the other hand, by Fermat's theorem

$$m^{p-1} \equiv 1 (\text{mod} p) \tag{29}$$

(28) and (29) yield $p = 2$. Thus, $p \neq 4k+3$.

Case 2: $p = 4k+1$. Let us consider (27) in mod4. $n^2 = 0$ or 1 in mod4. In both cases $n^{2p} = n^2 (\text{mod} 4)$. From (27) we get $n^2 \equiv m^2+n^2+1+1 (\text{mod} 4)$. Therefore, $m^2 \equiv -2 (\text{mod} 4)$, and again there is no solution.

Case 3: $p = 2$. The given equation is written as

$$n^4 - n^2 - 3 = m^2.$$

Let $l = n^2$. Readily, we do not get any solution for $l = 0, 1$. If $l = 4$, then there are four solutions: $(3,2,2), (-3,2,2), (3,-2,2), (-3,-2,2)$. There is no solution for $l > 4$, since in this case

$$(l-1)^2 = l^2 - 2l + 1 < m^2 = l^2 - l - 3 < l^2$$

Thus, (27) has four solutions: $(3,2,2), (-3,2,2), (3,-2,2), (-3,-2,2)$ and we are done. ■

NT5. Find all the positive integers x, y, z, t such that $2^x \cdot 3^y + 5^z = 7^t$.

Solution. Reducing modulo 3 we get $5^z \equiv 1$, therefore z is even, $z = 2c$, $c \in \mathbb{N}$.

Next we prove that t is even. Obviously, $t \geq 2$. Let us suppose that t is odd, $t = 2d+1$, $d \in \mathbb{N}$. The equation becomes $2^x \cdot 3^y + 25^c = 7 \cdot 49^d$.

If $x \geq 2$, reducing modulo 4, we get $1 \equiv 3$, contradiction.

For $x = 1$, we have $2 \cdot 3^y + 25^c = 7 \cdot 49^d$, and, reducing modulo 24, we obtain $2 \cdot 3^y + 1 \equiv 7 \Rightarrow 24 \mid 2(3^y - 3)$, i.e. $4 \mid 3^{y-1} - 1$, which means that $y - 1$ is even. Then, $y = 2b+1$, $b \in \mathbb{N}$.

We obtain $6 \cdot 9^b + 25^c = 7 \cdot 49^d$, and, reducing modulo 5, we get $(-1)^b \equiv 2 \cdot (-1)^d$, which is false, for all $b, d \in \mathbb{N}$. Hence t is even, $t = 2d$, $d \in \mathbb{N}$.

The equation can be written as $2^x \cdot 3^y + 25^c = 49^d \Leftrightarrow 2^x \cdot 3^y = \left(7^d - 5^c\right)\left(7^d + 5^c\right)$. As $\gcd\left(7^d - 5^c, 7^d + 5^c\right) = 2$ and $7^c + 5^c > 2$, there exist exactly three possibilities:(1) $\begin{cases} 7^d - 5^c = 2^{x-1} \\ 7^d + 5^c = 2 \cdot 3^y \end{cases}$; (2) $\begin{cases} 7^d - 5^c = 2 \cdot 3^y \\ 7^d + 5^c = 2^{x-1} \end{cases}$; (3) $\begin{cases} 7^d - 5^c = 2 \\ 7^d + 5^c = 2^{x-1} \cdot 3^y \end{cases}$

Case (1). We have $7^d = 2^{x-2} + 3^y$ and, reducing modulo 3, we get $2^{x-2} \equiv 1 \pmod 3$, hence $x - 2$ is even, i.e. $x = 2a + 2$, where $a \in \mathbb{N}^*$ since $a = 0$ would mean $3^y + 1 = 7^d$, which is impossible (even = odd).

We obtain $7^d - 5^c = 2 \cdot 4^a \overset{\text{mod}4}{\Longrightarrow} 7^d \equiv 1 \pmod 4 \Rightarrow d = 2e$, $e \in \mathbb{N}$. Then $49^e - 5^c = 2 \cdot 4^k \overset{\text{mod}8}{\Longrightarrow} 5^c \equiv 1 \pmod 8 \Rightarrow c = 2f$, $f \in \mathbb{N}$. We obtain $49^e - 25^f = 2 \cdot 4^a \overset{\text{mod}3}{\Longrightarrow} 0 \equiv 2 \pmod 3$, false. In conclusion, in this case there are no solutions to the equation.

Case (2). From $2^{x-1} = 7^d + 5^c \geq 12$, we obtain $x \geq 5$. Then $7^d + 5^c \equiv 0 \pmod 4$, i.e. $3^d + 1 \equiv 0 \pmod 4$, hence d is odd. As $7^d = 5^c + 2 \cdot 3^y \geq 11$, we get $d \geq 2$, hence $d = 2e + 1$, $e \in \mathbb{N}$.

As in the previous case, from $7^d = 2^{x-2} + 3^y$, reducing modulo 3, we obtain $x = 2a + 2$, with $a \geq 2$ (because $x \geq 5$). We get $7^d = 4^a + 3^y$, i.e. $7 \cdot 49^e = 4^a + 3^y$, hence, reducing modulo 8, we obtain $7 \equiv 3^y$, which is false, because 3^y is congruent mod8 either to 1 (if y is even) or to 3 (if y is odd). In conclusion, in this case there are no solutions to the equation.

Case (3). From $7^d = 5^c + 2$, it follows that the last digit of 7^d is 7, hence $d = 4k + 1$, $k \in \mathbb{N}$.

If $c \geq 2$, from $7^{4k+1} = 5^c + 2$, reducing modulo 25, we obtain $7 \equiv 2 \pmod{25}$, which is false.

For $c = 1$ we get $d = 1$, and the solution $x = 3$, $y = 1$, $z = t = 2$. ■

3 **NT6.** *If a, b, c, d are integers and $A = 2(a - 2b + c)^4 + 2(b - 2c + a)^4 + 2(c - 2a + b)^4$, $B = d(d+1)(d+2)(d+3) + 1$, prove that $\left(\sqrt{A} + 1\right)^2 + B$ cannot be a perfect square.*

Solution. First we prove the following Lemma

<u>Lemma</u>: *If x, y, z real numbers such that $x + y + z = 0$, then $2(x^4 + y^4 + z^4) = (x^2 + y^2 + z^2)^2$.*

Proof of the Lemma:

$$\begin{aligned} x^4 + y^4 + z^4 &= x^2x^2 + y^2y^2 + z^2z^2 = x^2(y+z)^2 + y^2(z+x)^2 + z^2(x+y)^2 \\ &= 2(x^2y^2 + y^2z^2 + z^2x^2) + 2xyz(x+y+z) \\ &= (x^2 + y^2 + z^2)^2 - x^4 - y^4 - z^4 \end{aligned}$$

and the claim follows.

Now back to our problem notice that $(a - 2b + c) + (b - 2c + a) + (c - 2a + b) = 0$, thus according to the lemma it holds

$$\begin{aligned} A &= 2(a - 2b + c)^4 + 2(b - 2c + a)^4 + 2(c - 2a + b)^4 \\ &= \left[(a - 2b + c)^2 + (b - 2c + a)^2 + (c - 2a + b)^2\right] = \left[6(a^2 + b^2 + c^2 - ab - bc - ca)\right]^2 \end{aligned}$$

Since $a^2+b^2+c^2 \geq ab+bc+ca$ we have that

$$\sqrt{A}+1 = 6(a^2+b^2+c^2-ab-bc-ca)+1$$

In addition, it is easy to check that

$$B = d(d+1)(d+2)(d+3) = (d^2+3d+1)^2$$

Let us set $6(a^2+b^2+c^2-ab-bc-ca)+1 = m$, $d^2+3d+1 = n$. We need to prove that the number $\left(\sqrt{A}+1\right)^2 + B = m^2+n^2$ is not a perfect square.

Since both m, n are odd integers, both m^2, n^2 are integers of the form $4k+1$, so the number m^2+n^2 is an integer of the form $4k+2$. But it is well known that all perfect squares are of the form $4k$ or $4k+1$, and we are done. ■

NT7. *Find all natural numbers a, b, c for which $1997^a + 15^b = 2012^c$.*

Solution. $1997^a + 15^b = 2012^c \Rightarrow 1+(-1)^b \equiv 0 (\text{mod} 4)$, so b is an odd number.
$1997^a + 15^b = 2012^c \Rightarrow 1+0 \equiv 2^c (\text{mod} 3)$, so c is even, say $c = 2c_1$.

We intend to consider the given equation modulo 8 and for this reason we discern two cases:

(1): $c = 1$. Clearly then $a = b = 1$ and $a = b = c = 1$ is a solution. This is actually the only solution of the given equation since in the remaining case where $c > 1$ it will be shown that there exist no solution.

(2): $c > 1$. Then $2012^c = (4 \cdot 503)^c$ is a multiple of 8 and
$1997^a + 15^b = 2012^c \Rightarrow 5^a + (-1)^b \equiv 5^a + (-1) \equiv 0 (\text{mod} 8)$, so a is even, say $a = 2a_1$. Hence

$$3^b \cdot 5^b = 15^b = 2012^c - 1997^a = (2012^{c_1} - 1997^{a_1}) \cdot (2012^{c_1} + 1997^{a_1}).$$

Observe that $2012^{c_1} - 1997^{a_1}, 2012^{c_1} + 1997^{a_1}$ are both greater than 1 and prime to each other as $gcd(2012^{c_1} - 1997^{a_1}, 2012^{c_1} + 1997^{a_1}) = gcd(2012^{c_1} - 1997^{a_1}, 2 \cdot 1997^{a_1}) = 1$. So there exist two cases:

$$\text{Case 1:} \begin{array}{l} 2012^{c_1} - 1997^{a_1} = 5^b \\ 2012^{c_1} + 1997^{a_1} = 3^b \end{array}, \quad \text{Case 2:} \begin{array}{l} 2012^{c_1} - 1997^{a_1} = 3^b \\ 2012^{c_1} + 1997^{a_1} = 5^b \end{array}$$

Case 1:

$$\begin{array}{c} 2012^{c_1} - 1997^{a_1} = 5^b \Rightarrow 2^{c_1} - 2^{a_1} \equiv 0 (\text{mod} 5) \Rightarrow c_1 \equiv a_1 (\text{mod} 5) \\ 2012^{c_1} + 1997^{a_1} = 3^b \Rightarrow 2^{c_1} + 2^{a_1} \equiv 0 (\text{mod} 5) \Rightarrow c_1 \equiv a_1 + 1 (\text{mod} 5) \end{array}$$

a contradiction.

Case 2:

$$\begin{array}{c} 2012^{c_1} - 1997^{a_1} = 3^b \\ 2012^{c_1} + 1997^{a_1} = 5^b \end{array}$$

Since b is an odd number we get $2012^{c_1} + 1997^{a_1} \equiv 5^b (\text{mod} 3) \Rightarrow 2^{c_1} + 2^{a_1} \equiv 5^b \equiv 2 (\text{mod} 3)$ so a_1, c_1 are even numbers, say $a_1 = 2a_2, c_1 = 2c_2$. Then

$$(2012^{c_2} - 1997^{a_2}) \cdot (2012^{c_2} + 1997^{a_2}) = 3^b$$

But $gcd(2012^{c_2} - 1997^{a_2}, 2012^{c_2} + 1997^{a_2}) = 1$ and the above implies $2012^{c_2} - 1997^{a_2} = 1$. But then mod4, we get $0 - 1 \equiv (\text{mod}4)$, a contradiction.

Therefore there exists no solution for $c > 1$.

Hence $a = b = c = 1$ is the only solution. ∎