1. Determine the polygons with n sides, $n \geq 4$, not necessarily convex, that satisfy the property that the reflection of every vertex of polygon with respect to every diagonal of the polygon does not fall outside the polygon.

Note. A diagonal is any segment joining two non-neighbouring vertices of the polygon; the reflection is considered with respect to the support line of the diagonal.

Solution. A polygon with this property has to be convex, otherwise, we consider an edge of the convex hull of the set of the vertices that is not an edge of the polygon. All the other vertices are situated in one of the half-planes determined by the support-line of this edge, therefore the reflection of the other vertices falls outside the polygonal.

Now we fix a diagonal. It divides the polygon into two parts p_1, p_2. The reflection of p_1 falls into the interior of p_2 and vice versa. As a consequence, the diagonal is a symmetry axis for the polygon. Then every diagonal of the polygon bisects the angles of the polygonal and this means that there are 4 vertices and the polygon is a rhombus.
Any rhombus satisfies the desired condition.

2. Decide if it is possible to consider 2011 points in a plane such that:

i) the distance between every two of these points is different from 1, and

ii) every unit circle centered at one of these points leaves exactly 1005 of the points outside the circle.

Solution. It is not possible. If such a configuration would exist, the number of segments starting from any of the 2001 points towards the other ones and having length less then 1 would be 1005. Their total number would be $1005 \cdot 2011$. But each segment is counted twice, while $1005 \cdot 2011$ is odd, false.

3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying

$$\sum_{k=1}^{n-1} \min(x_k, x_{k+1}) = \min(x_1, x_n).$$

Show that

$$\sum_{k=2}^{n-1} x_k \geqslant 0.$$

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Solution. Since $\min(a, b) = \frac{1}{2}(a + b - |a - b|)$, we have

$$\sum_{k=1}^{n-1} \frac{1}{2}(x_k + x_{k+1} - |x_k - x_{k+1}|) = \frac{1}{2}(x_1 + x_n - |x_1 - x_n|) \Leftrightarrow \dots$$

$$2(x_2 + \ldots + x_{n-1}) + |x_1 - x_n| = |x_1 - x_2| + \cdots + |x_{n-1} - x_n|.$$

As $|x_1 - x_2| + \cdots + |x_{n-1} - x_n| \geqslant |x_1 - x_2 + \cdots + x_{n-1} - x_n| = |x_1 - x_n|$, we obtain the desired conclusion.