ON A HOMOGENEOUS INEQUALITY GIVEN AT THE 2004 J.B.M.O.

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Abstract. Starting from a homogeneous JBMO inequality, other interesting related results are obtained.

Keywords: elementary inequality
MSC : 26D05

1. Introduction

The first problem given to the Junior Balkan Mathematical Olympiad in 2004, proposed by Albania, see [2], [3], was the following:

Problem 1. *Prove that the inequality*:

$$\frac{x+y}{x^2-xy+y^2} \leq \frac{2\sqrt{2}}{\sqrt{x^2+y^2}}, \tag{1}$$

holds for all real numbers x *and* y*, not both equal to* 0.

In the very recent book of S. Bilchev [2], two solutions of this problem are given. We present here both of them, in view of some comments and

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developments[1)].

First Solution. If $x + y \leq 0$, the inequality obviously holds (the left hand side is negative or zero, while the right hand side is positive).

It is also easy to check that for $x = 0$ or $y = 0$, the inequality in (1) is strict.

Consider, therefore, only the case $x+y > 0$ and $x \neq 0$, $y \neq 0$. Then (1) can be equivalently written under the form:

$$(x+y) \cdot \frac{x^2+y^2}{2} \leq \sqrt{2(x^2+y^2)} \cdot (x^2 - xy + y^2). \tag{2}$$

But

$$x + y \leq \sqrt{2(x^2+y^2)} \quad (\Leftrightarrow (x-y)^2 \geq 0) \tag{3}$$

and

$$\frac{x^2+y^2}{2} \leq x^2 - xy + y^2 \quad (\Leftrightarrow (x-y)^2 \geq 0)\,. \tag{4}$$

Now, in view of the fact that the numbers in both sides of (3) and (4) are positive, by multiplying (3) and (4) side by side we get exactly (2).

Equality in (1) holds if and only if equality holds in (3) and (4), that is, if $x = y$.

Second Solution. Similarly to the first solution, consider only the case $x + y > 0$ and $x \neq 0$, $y \neq 0$. By denoting $S = x^2 + y^2$ and $P = xy$, the inequality (1) can be equivalently written, after squaring both sides, as:

$$(S + 2P) \cdot S \leq 8(S - P)^2$$

which reduces to $7S^2 - 18SP + 8P^2 \geq 0 \quad \Leftrightarrow (S-2P)(7S-4P) \geq 0$. But $S - 2P = (x-y)^2 \geq 0$, with equality if and only if $x = y$, and

$$7S - 4P = 7x^2 - 4xy + 7y^2 = y^2 \cdot \left[7\left(\frac{x}{y}\right)^2 - 4\frac{x}{y} + 7\right] > 0,$$

for all $x, y \in \mathbb{R}, x \neq 0, y \neq 0$.

2. Other similar inequalities

Now let us have a close look on the main argument in the second solution presented above.

Basically, the key tool in proving inequality (1) was to reduce it to an inequality of the form:

$$(S - 2P)(aS + bP) \geq 0, \tag{5}$$

where a and b were some constants for which $aS + bP > 0$, for all $x, y \in \mathbb{R}$, $x \neq 0$, $y \neq 0$. As $\operatorname{sgn}(aS + bP) = \operatorname{sgn}(at^2 + bt + a)$, $t \in \mathbb{R}$, all that is needed in order to have (5) satisfied is to have: $at^2 + bt + a > 0$, $\forall t \in \mathbb{R}$. But, in view of the properties a the quadratic function, this happens if:

[1)]Dedicated to the memory of my friend Professor *Svetoslav Jordanov Bilchev* (1946-2010), former Dean of Faculty of Education and Head of Department of Algebra and Geometry, University of Rousse, Bulgaria

1) $a > 0$ and 2) $\Delta = b^2 - 4a^2 < 0$,
that is, if $a > 0$ and $b \in (-2a, 2a)$. In that case (5) can be equivalently written as:

$$(a+1)(S-P)^2 \geq (a+2b+1)P^2 + S^2 - (b+2)SP. \tag{6}$$

Now, in order to get the factor S in (6), we must have:

$$a + 2b + 1 = 0, \tag{7}$$

which was satisfied in the original case of Problem 1, when we have had $a = 7$ and $b = -4$.

It is easy to see that if we would like to get:

$$S^2 - (b+2)SP = S(S-(b+2)P) = S(S+2P),$$

then we must have $-b+2 = 2$, i.e., $b = -4$ and then $a = 7$, which gives exactly the original inequality.

But even though, for other values of a and b, we are not able to obtain precisely the expression $S+2P$, i.e., a perfect square, we still find interesting and not trivial inequalities.

1. If we take $a = 5$, then by by (7) we get $b = -3$ and obtain the following new inequality:

Problem 2. *Prove that the inequality:*

$$\frac{\sqrt{x^2+xy+y^2}}{x^2-xy+y^2} \leq \frac{\sqrt{6}}{\sqrt{x^2+y^2}}, \tag{8}$$

holds for all real numbers x and y, not both equal to 0.

2. If we take $a = 9$, then by by (7) we get $b = -5$ and obtain the following new inequality:

Problem 3. *Prove that the inequality:*

$$\frac{\sqrt{x^2+3xy+y^2}}{x^2-xy+y^2} \leq \frac{\sqrt{10}}{\sqrt{x^2+y^2}}, \tag{9}$$

holds for all real numbers x and y, not both equal to 0.

3. If we take $a = 11$, then by by (7) we get $b = -6$ and obtain the following new inequality:

Problem 4. *Prove that the inequality:*

$$\frac{\sqrt{x^2+4xy+y^2}}{x^2-xy+y^2} \leq \frac{2\sqrt{3}}{\sqrt{x^2+y^2}}, \tag{10}$$

holds for all real numbers x and y, not both equal to 0.

4. If we take $a = 15$, then by by (7) we get $b = -8$ and obtain the following new inequality:

Problem 5. *Prove that the inequality*:

$$\frac{\sqrt{x^2+6xy+y^2}}{x^2-xy+y^2} \leq \frac{4}{\sqrt{x^2+y^2}}, \tag{11}$$

holds for all real numbers x and y, not both equal to 0.

In the end of this note, notice that there is an essential difference between the first and the second solution of Problem 1: while the later opened a door for further investigations, the former did not.

This is the reason why we can call a solution like the second one, as a *creative* solution, see [1].

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This is a modest homage to *Slavy*'s special interest in *Gazeta Matematică*, a journal which he regularly read and used in his scientific and teaching activity.

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GENERALIZĂRI ALE UNOR FORMULE TRIGONOMETRICE

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Abstract. This article establishes formulae for the sine, the cosine and the tangent of a sum of several reals and gives a sufficient condition for their validity

Keywords: domain of the tangent and applications

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În prima parte a acestei note, ne-am propus să generalizăm formulele de calcul a cosinusului şi sinusului sumei a două numere reale, precum şi formulele de transformare a produsului de două cosinusuri, respectiv două sinusuri în sume.

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