0.1 Number Theory

NT1 Solve in positive integers the equation $1005^x + 2011^y = 1006^z$.

Solution

We have $1006^z > 2011^y > 2011$, hence $z \geq 2$. Then $1005^x + 2011^y \equiv 0 \ (mod\ 4)$.
But $1005^x \equiv 1 \ (mod\ 4)$, so $2011^y \equiv -1 \ (mod\ 4) \Rightarrow y$ is odd, i.e. $2011^y \equiv -1 \ (mod\ 1006)$.
Since $1005^x + 2011^y \equiv 0 \ (mod\ 1006)$, we get $1005^x \equiv 1 \ (mod\ 1006) \Rightarrow x$ is even.
Now $1005^x \equiv 1 \ (mod\ 8)$ and $2011^y \equiv 3 \ (mod\ 8)$, hence $1006^z \equiv 4 \ (mod\ 8) \Rightarrow z = 2$.
It follows that $y < 2 \Rightarrow y = 1$ and $x = 2$. The solution is $(x, y, z) = (2, 1, 2)$.

NT2 Find all prime numbers p such that there exist positive integers x, y that satisfy the relation $x(y^2 - p) + y(x^2 - p) = 5p$.

Solution

The given equation is equivalent to $(x+y)(xy-p) = 5p$. Obviously $x + y \geq 2$.
We will consider the following three cases:
Case 1: $x + y = 5$ and $xy = 2p$. The equation $x^2 - 5x + 2p = 0$ has at least a solution, so we must have $0 \leq \Delta = 25 - 8p$ which implies $p \in \{2, 3\}$. For $p = 2$ we obtain the solutions $(1, 4)$ and $(4, 1)$ and for $p = 3$ we obtain the solutions $(2, 3)$ and $(3, 2)$.
Case 2: $x + y = p$ and $xy = p + 5$. We have $xy - x - y = 5$ or $(x-1)(y-1) = 6$ which implies $(x, y) \in \{(2, 7); (3, 4); (4, 3); (7, 2)\}$. Since p is prime, we get $p = 7$.
Case 3: $x + y = 5p$ and $xy = p + 1$. We have $(x-1)(y-1) = xy - x - y + 1 = 2 - 4p < 0$. But this is impossible since $x, y \geq 1$.
Finally, the equation has solutions in positive integers only for $p \in \{2, 3, 7\}$.

NT3 Find all positive integers n such that the equation $y^2 + xy + 3x = n(x^2 + xy + 3y)$ has at least a solution (x, y) in positive integers.

Solution

Clearly for $n = 1$, each pair (x, y) with $x = y$ is a solution. Now, suppose that $n > 1$ which implies $x \neq y$. We have $0 < n - 1 = \dfrac{y^2 + xy + 3x}{x^2 + xy + 3y} - 1 = \dfrac{(x+y-3)(y-x)}{x^2 + xy + 3y}$.
Since $x + y \geq 3$, we conclude that $x + y > 3$ and $y > x$. Take $d = gcd(x + y - 3; x^2 + xy + 3y)$. Then d divides $x^2 + xy + 3y - x(x + y - 3) = 3(x + y)$. Then d also divides $3(x + y) - 3(x + y - 3) = 9$, hence $d \in \{1, 3, 9\}$. As $n - 1 = \dfrac{\frac{x+y-3}{d}(y-x)}{\frac{x^2+xy+3y}{d}}$ and $gcd\left(\dfrac{x+y-3}{d}; \dfrac{x^2+xy+3y}{d}\right) = 1$, it follows that $\dfrac{x^2+xy+3y}{d}$ divides $y - x$, which leads to $x^2 + xy + 3y \leq dy - dx \Leftrightarrow x^2 + dx \leq (d - 3 - x)y$. It is necessary that

$d-3-x>0 \Rightarrow d>3$, therefore $d=9$ and $x<6$. Take $x+y-3=9k,\ k\in\mathbb{N}^*$ since $d \mid x+y-3$ and we get $y=9k+3-x$. Hence $n-1=\dfrac{k(9k+3-2x)}{k(x+3)+1}$. Because k and $k(x+3)+1$ are relatively prime, the number $t=\dfrac{9k+3-2x}{k(x+3)+1}$ must be integer for some positive integers $x<6$. It remains to consider these values of x:

1) For $x=1$, then $t=\dfrac{9k+1}{4k+1}$ and since $1<t<3$, we get $t=2,\ k=1,\ y=11$, so $n=3$.

2) For $x=2$, then $t=\dfrac{9k-1}{5k+1}$ and since $1<t<2$, there are no solutions in this case.

3) For $x=3$, then $t=\dfrac{9k-3}{6k+1}$ and since $1\neq t<2$, there are no solutions in this case.

4) For $x=4$, then $t=\dfrac{9k-5}{7k+1}<2$,i.e. $t=1$ which leads to $k=3,\ y=26$, so $n=4$.

5) For $x=5$, then $t=\dfrac{9k-7}{8k+1}<2$,i.e. $t=1$ which leads to $k=8,\ y=70$, so $n=9$.

Finally, the answer is $n\in\{1,3,4,9\}$.

NT4 Find all prime positive integers p,q such that $2p^3-q^2=2(p+q)^2$.

Solution 1

The given equation can be rewritten as $2p^2(p-1)=q(3q+4p)$.

Hence $p \mid 3q^2+4pq \Rightarrow p \mid 3q^2 \Rightarrow p \mid 3q$ (since p is a prime number) $\Rightarrow p \mid 3$ or $p \mid q$.

If $p \mid q$, then $p=q$. The equation becomes $2p^3-9p^2=0$ which has no prime solution.

If $p \mid 3$, then $p=3$. The equation becomes $q^2+4q-12=0 \Leftrightarrow (q-2)(q+6)=0$.

Since $q>0$, we get $q=2$, so we have the solution $(p,q)=(3,2)$.

Solution 2

Since $2p^3$ and $2(p+q^2)$ are even, q^2 is also even, thus $q=2$ because it is a prime number.

The equation becomes $p^3-p^2-4p-6=0 \Leftrightarrow (p^2-4)(p-1)=10$.

If $p\geq 4$, then $(p^2-4)(p-1)\geq 12\cdot 3>10$, so $p\leq 3$. A direct verification gives $p=3$.

Finally, the unique solution is $(p,q)=(3,2)$.

NT5 Find the least positive integer such that the sum of its digits is 2011 and the product of its digits is a power of 6.

Solution

Denote this number by N. Then N can not contain the digits $0,5,7$ and its digits must be written in increasing order. Suppose that N has x_1 ones, x_2 twos, x_3 threes, x_4 fours, x_6 sixes, x_8 eights and x_9 nines, then $x_1+2x_2+3x_3+4x_4+6x_6+8x_8+9x_9=2011$. ***(1)***

The product of digits of the number N is a power of 6 when we have the relation $x_2+2x_4+x_6+3x_8=x_3+x_6+2x_9$, hence $x_2-x_3+2x_4+3x_8-2x_9=0$. ***(2)***

Denote by S the number of digits of N ($S = x_1 + x_2 + ... + x_9$). In order to make the coefficients of x_8 and x_9 equal, we multiply relation ***(1)*** by 5, then we add relation ***(2)***. We get $43x_9 + 43x_8 + 30x_6 + 22x_4 + 14x_3 + 11x_2 + 5x_1 = 10055 \Leftrightarrow 43S = 10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$. Then $10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$ is a multiple of 43 not less than 10055. The least such number is 10062, but the relation $10062 = 10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$ means that among $x_1, x_2, ..., x_6$ there is at least one positive, so $10062 = 10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1 \geq 10055 + 13 = 10068$ which is obviously false. The next multiple of 43 is 10105 and from relation $10105 = 10055 + 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$ we get $50 = 13x_6 + 21x_4 + 29x_3 + 32x_2 + 38x_1$. By writing as $13x_6 + 21(x_4 - 1) + 29(x_3 - 1) + 32x_2 + 38x_1 = 0$, we can easily see that the only possibility is $x_1 = x_2 = x_6 = 0$ and $x_3 = x_4 = 1$. Then $S = 235,\ x_8 = 93,\ x_9 = 140$. Since S is strictly minimal, we conclude that $N = 34\underbrace{88...8}_{93}\underbrace{99...9}_{140}$.