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18th JUNIOR BALKAN MATHEMATICAL OLYMPIAD

June 21-26 ▪ Ohrid ▪ R. Macedonia

Problem Shortlist with solutions

UNION OF MATHEMATICIANS
OF MACEDONIA

18 Junior Balkan Mathematical Olympiad
June, 21-26 Ohrid

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with solutions

[redacted]

These Shortlist Problems have been kept strictly confidential until JBMO 2015

Contributing countries

The organizing committee and the Problem Selection Committee of JBMO thank the following countries for submitting problems:

Albania, Bulgaria , Cyprus, Greece, Serbia, Romania, Turkey, Bosna and Herzegovina, Montenegro, France, Tadjikistan

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Algebra

A1

For any real number a, let $\lfloor a \rfloor$ denote the greatest integer not exceeding a. In positive real numbers solve the following equation

$$n+\left\lfloor \sqrt{n} \right\rfloor+\left\lfloor \sqrt[3]{n} \right\rfloor=2014.$$

A2

Let a, b and c be positive real numbers such that $abc=\frac{1}{8}$. Prove the inequality

$$a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2 \geq \frac{15}{16}.$$

When does equality hold?

A3

Let a,b,c be positive real numbers such that $abc=1$. Prove that:

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \geq 3(a+b+c+1).$$

When does equality hold?

A4

Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{7+2b}{1+a}+\frac{7+2c}{1+b}+\frac{7+2a}{1+c} \geq \frac{69}{4}.$$

When does equality hold?

A5

Let x,y,z be non-negative real numbers satisfying $x+y+z=xyz$. Prove that

$$2\left(x^2+y^2+z^2\right) \geq 3(x+y+z),$$

and determine when equality occurs.

A6

Let a,b,c be positive real numbers. Prove that

$$\left((3a^2+1)^2+2\left(1+\frac{3}{b}\right)^2\right)\left((3b^2+1)^2+2\left(1+\frac{3}{c}\right)^2\right)\left((3c^2+1)^2+2\left(1+\frac{3}{a}\right)^2\right)\geq 48^3 .$$

When does equality hold?

A7

Let a,b,c be positive real numbers such that $a^2+b^2+c^2=48$. Prove

$$a^2\sqrt{2b^3+16}+b^2\sqrt{2c^3+16}+c^2\sqrt{2a^3+16}\leq 24^2.$$

When does equality hold?

A8

Let x, y and z be positive real numbers such that $xyz=1$. Prove the inequality

$$\frac{1}{x(ay+b)}+\frac{1}{y(az+b)}+\frac{1}{z(ax+b)}\geq 3, \text{ if:}$$

a) $a=0$ *and* $b=1$; *b)* $a=1$ *and* $b=0$; *c)* $a+b=1$ *for* $a,b>0$

When does the equality hold true?

Remark. The problem can be reformulated:

Let a, b, x, y and z be nonnegative real numbers such that $xyz=1$ and $a+b=1$. Prove the inequality

$$\frac{1}{x(ay+b)}+\frac{1}{y(az+b)}+\frac{1}{z(ax+b)}\geq 3.$$

When does the equality hold true?

A9

Let n be a positive integer, and let $x_1,\ldots,x_n,y_1,\ldots,y_n$ be positive real numbers such that $x_1+\ldots+x_n=y_1+\ldots+y_n=1$. Show that

$$|x_1-y_1|+\ldots|x_n-y_n|\leq 2-\min_{1\leq i\leq n}\frac{x_i}{y_i}-\min_{1\leq i\leq n}\frac{y_i}{x_i} .$$

Combinatorics

C1

Several (at least two) segments are drawn on a board. Select two of them, and let a and b be their lengths. Delete the selected segments and draw a segment of length $\frac{ab}{a+b}$. Continue this procedure until only one segment remains on the board. Prove:

a) the length of the last remaining segment does not depend on the order of the deletions.

b) for every positive integer n, the initial segments on the board can be chosen with

C2

In a country with n cities, all direct airlines are two-way. There are $r > 2014$ routes between pairs of different cities that include no more than one intermediate stop (the direction of each route matters). Find the least possible n and the least possible r for that value of n.

C3

For a given positive integer n, two players A and B play the following game: Given is pile of a stones. The players take turn alternatively with A going first. On each turn the player is allowed to take one stone, a prime number of stones, or a multiple of n stones. The winner is the one who takes the last stone. Assuming perfect play, find the number of values for a, for which A cannot win.

C4

Let $A = 1 \cdot 4 \cdot 7 \cdot \ldots \cdot 2014$ be the product of the numbers less or equal to 2014 that give remainder 1 when divided by 3. Find the last non-zero digit of A.

Geometry

G1

Let ABC *be a triangle with* $\measuredangle B = \measuredangle C = 40^{\circ}$. *The bisector of the* $\measuredangle B$ *meets* AC *at the point* D. *Prove that* $\overline{BD} + \overline{DA} = \overline{BC}$.

G2

Let ABC *be an acute triangle with* $\overline{AB} < \overline{AC} < \overline{BC}$ *and* $c(O,R)$ *be its circumcircle. Denote with* D *and* E *be the points diametrically opposite to the points* B *and* C, *respectively. The circle* $c_1\left(A,\overline{AE}\right)$ *intersects* $\overline{AC}$ *at point* K, *the circle* $c_2\left(A,\overline{AD}\right)$ *intersects* BA *at point* L *(* A *lies between* B *and* L *). Prove that the lines* EK *and* DL *meet on the circle* c.

G3

Let $CD \perp AB$ $(D \in AB)$, $DM \perp AC$ $(M \in AC)$ *and* $DN \perp BC$ $(N \in BC)$ *for an acute triangle* ABC *with area* S. *If* H_1 *and* H_2 *are the orthocentres of the triangles* MNC *and* MND *respectively. Evaluate the area of the quadrilateral* AH_1BH_2.

G4

Let ABC *be a triangle such that* $\overline{AB} \neq \overline{AC}$. *Let* M *be a midpoint of* $\overline{BC}$, H *the orthocenter of* ABC, O_1 *the midpoint of* $\overline{AH}$ *and* O_2 *the circumcenter of* BCH. *Prove that* O_1AMO_2 *is a parallelogram.*

G5

Let ABC *be a triangle with* $\overline{AB} \neq \overline{BC}$, *and let* BD *be the internal bisector of* $\measuredangle ABC\,(D \in AC)$. *Denote the midpoint of the arc* AC *which contains point* B *by* M. *The circumcircle of the triangle* BDM *intersects the segment* AB *at point* $K \neq B$, *and let* J *be the reflection of* A *with respect to* K. *If* $DJ \cap AM = \{O\}$, *prove that the points* J, B, M, O *belong to the same circle.*

G6

Let $ABCD$ be a quadrilateral whose sides AB and CD are not parallel, and let O be the intersection of its diagonals. Denote with H_1 and H_2 the orthocenters of the triangles OAB and OCD, respectively. If M and N are the midpoints of the segments $\overline{AB}$ and $\overline{CD}$, respectively, prove that the lines MN and H_1H_2 are parallel if and only if $\overline{AC} = \overline{BD}$.

Number theory

N1

Each letter of the word OHRID corresponds to a different digit belonging to the set {1,2,3,4,5}.

Decipher the equality $(O+H+R+I+D)^2:(O-H-R+I+D)=O^{H^{R^{I^{D}}}}$.

N2

Find all triples (p,q,r) *of distinct primes* p , q *and* r *such that*

$$3p^4-5q^4-4r^2=26.$$

N3

Find the integer solutions of the equation

$$x^2=y^2(x+y^4+2y^2).$$

N4

Prove there are no integers a *and* b *satisfying the following conditions:*

i) $16a-9b$ *is a prime number*

ii) ab *is a perfect square*

iii) $a+b$ *is a perfect square*

N5

Find all nonnegative integers x, y, z such that

$$2013^x+2014^y=2015^z.$$

N6

Vukasin, Dimitrije, Dusan, Stefan and Filip asked their professor to guess a three consecutive positive integer numbers after they had told him these (true) sentences:

Vukasin: "Sum of the digits of one of them is a prime number. Sum of the digits of some of the other two is an even perfect number (n is perfect if $\sigma(n) = 2n$). Sum of the digits of the remaining number is equal to the number of its positive divisors."

Dimitrije:"Each of these three numbers has no more than two digits 1 in its decimal representation."

Dusan:"If we add 11 to one of them, we obtain a square of an integer."

Stefan:"Each of them has exactly one prime divisor less then 10."

Filip:"The 3 numbers are square-free."

Their professor gave the correct answer. Which numbers did he say?

A1

For any real number a, let $\lfloor a \rfloor$ denote the greatest integer not exceeding a. In positive real numbers solve the following equation

$$n+\lfloor \sqrt{n} \rfloor+\lfloor \sqrt[3]{n} \rfloor=2014.$$

Solution1. Obviously n must be positive integer. Now note that $44^2=1936<2014<2025=45^2$ and $12^3<1900<2014<13^3$.

If $n<1950$ than $2014=n+\lfloor \sqrt{n} \rfloor+\lfloor \sqrt[3]{n} \rfloor<1950+44+12=2006$, a contradiction!

So $n\geq 1950$. Also if $n>2000$ than $2014=n+\lfloor \sqrt{n} \rfloor+\lfloor \sqrt[3]{n} \rfloor>2000+44+12=2056$, a contradiction!

So $1950\leq n\leq 2000$, therefore $\lfloor \sqrt{n} \rfloor=44$ and $\lfloor \sqrt[3]{n} \rfloor=12$. Plugging that into the original equation we get:

$$n+\lfloor \sqrt{n} \rfloor+\lfloor \sqrt[3]{n} \rfloor=n+44+12=2014$$

From which we get $n=1956$, which is the only solution.

Solution2. Obviously n must be positive integer. Since $n\leq 2014$, $\sqrt{n}<45$ and $\sqrt[3]{n}<13$.

Form $n=2014-\lfloor \sqrt{n} \rfloor-\lfloor \sqrt[3]{n} \rfloor>2014-45-13=1956$, $\sqrt{n}>44$ and $\sqrt[3]{n}>12$, thus $\lfloor \sqrt{n} \rfloor=44$ and $\lfloor \sqrt[3]{n} \rfloor=12$ and $n=2014-\lfloor \sqrt{n} \rfloor-\lfloor \sqrt[3]{n} \rfloor=2014-44-12=1958$.

(Very Easy)

A2

Let a, b and c be positive real numbers such that $abc=\frac{1}{8}$. Prove the inequality

$$a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2\geq\frac{15}{16}.$$

When does equality hold?

Solution1. By using The Arithmetic-Geometric Mean Inequality for 15 positive numbers, we find that

$$\begin{aligned}&a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2=\\&=\frac{a^2}{4}+\frac{a^2}{4}+\frac{a^2}{4}+\frac{a^2}{4}+\frac{b^2}{4}+\frac{b^2}{4}+\frac{b^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}+\frac{c^2}{4}+\frac{c^2}{4}+\frac{c^2}{4}+a^2b^2+b^2c^2+c^2a^2\geq\\&\geq 15\sqrt[15]{\frac{a^{12}b^{12}c^{12}}{4^{12}}}=15\sqrt[5]{\left(\frac{abc}{4}\right)^4}=15\sqrt[5]{\left(\frac{1}{32}\right)^4}=\frac{15}{16}\end{aligned}$$

as desired. Equality holds if and only if $a=b=c=\frac{1}{2}$.

Solution2. By using AM-GM we obtain

$$\begin{aligned}&\left(a^2+b^2+c^2\right)+\left(a^2b^2+b^2c^2+c^2a^2\right)\geq 3\sqrt[3]{a^2b^2c^2}+3\sqrt[3]{a^4b^4c^4}=\\&=3\sqrt[3]{\left(\frac{1}{8}\right)^2}+3\sqrt[3]{\left(\frac{1}{8}\right)^4}=\frac{3}{4}+\frac{3}{16}=\frac{15}{16}.\end{aligned}$$

The equality holds when $a^2=b^2=c^2$, i.e. $a=b=c=\frac{1}{2}$.

A3

Let a,b,c be positive real numbers such that $abc=1$. Prove that:

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq 3(a+b+c+1).$$

When does equality hold?

Solution1. By using AM-GM ($x^2+y^2+z^2\geq xy+yz+zx$) we have

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)+\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)+\left(c+\frac{1}{a}\right)\left(a+\frac{1}{b}\right)$$

$$=\left(ab+1+\frac{a}{c}+a\right)+\left(bc+1+\frac{b}{a}+b\right)+\left(ca+1+\frac{c}{b}+c\right)$$

$$=ab+bc+ca+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}+3+a+b+c.$$

Notice that by AM-GM we have $ab+\frac{b}{a}\geq 2b$, $bc+\frac{c}{b}\geq 2c$, and $ca+\frac{a}{c}\geq 2a$.

Thus ,

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\left(ab+\frac{b}{a}\right)+\left(bc+\frac{c}{b}\right)+\left(ca+\frac{a}{c}\right)+3+a+b+c\geq 3(a+b+c+1).$$

The equality holds if and only if $a=b=c=1$.

Solution2. From QM-AM we obtain

$$\sqrt{\frac{(a+\frac{1}{b})^2+(b+\frac{1}{c})^2+(c+\frac{1}{a})^2}{3}}\geq\frac{a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a}}{3}\Leftrightarrow$$

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\frac{(a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a})^2}{3} \qquad (1)$$

From AM-GM we have $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3\sqrt[3]{\frac{1}{abc}}=3$, and substituting in (1) we get

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq\frac{(a+\frac{1}{b}+b+\frac{1}{c}+c+\frac{1}{a})^2}{3}\geq\frac{(a+b+c+3)^2}{3}=$$

$$=\frac{(a+b+c)(a+b+c)+6(a+b+c)+9}{3}\geq\frac{(a+b+c)3\sqrt[3]{abc}+6(a+b+c)+9}{3}=$$

$$=\frac{9(a+b+c)+9}{3}=3(a+b+c+1)$$

The equality holds if and only if $a=b=c=1$.

A4

Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{7+2b}{1+a}+\frac{7+2c}{1+b}+\frac{7+2a}{1+c}\geq\frac{69}{4}.$$

When does equality hold?

Solution1. The inequality can be written as: $\frac{5+2(1+b)}{1+a}+\frac{5+2(1+c)}{1+b}+\frac{5+2(1+a)}{1+c}\geq\frac{69}{4}$.

We substitute $1+a=x, 1+b=y, 1+c=z$.

So, we have to prove the inequality

$$\frac{5+2y}{x}+\frac{5+2z}{y}+\frac{5+2x}{z}\geq\frac{69}{4}\Leftrightarrow 5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+2\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)\geq\frac{69}{4}$$

where $x,y,z>1$ real numbers and $x+y+z=4$.

We have

- $\frac{x+y+z}{3}\geq\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{9}{x+y+z}\Leftrightarrow\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{9}{4}$
- $\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\geq 3\cdot\sqrt[3]{\frac{y}{x}.\frac{z}{y}.\frac{x}{z}}=3$

Thus, $\frac{5+2y}{x}+\frac{5+2z}{y}+\frac{5+2x}{z}=5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+2\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)\geq 5\cdot\frac{9}{4}+2\cdot 3=\frac{69}{4}$.

The equality holds, when $\left(x=y=z, \frac{y}{x}=\frac{z}{y}=\frac{x}{z}, x+y+z=4\right)$, thus $x=y=z=\frac{4}{3}$, i.e. $a=b=c=\frac{1}{3}$.

('Εξω)
~~A5~~

Let x, y, z be non-negative real numbers satisfying $x+y+z=xyz$. Prove that

$$2(x^2+y^2+z^2)\geq 3(x+y+z),$$

and determine when equality occurs.

Solution. Equality holds when $x=y=z=0$.

Apply AM-GM to $x+y+z=xyz$,

$$xyz=x+y+z\geq 3\sqrt[3]{xyz}\Rightarrow (xyz)^3\geq \left(3\sqrt[3]{xyz}\right)^3$$

$$\Rightarrow x^3y^3z^3\geq 27xyz$$

$$\Rightarrow x^2y^2z^2\geq 27$$

$$\Rightarrow \sqrt[3]{x^2y^2z^2}\geq 3$$

Also by AM-GM we have, $x^2+y^2+z^2\geq 3\sqrt[3]{x^2y^2z^2}\geq 9$.

Therefore we get $x^2+y^2+z^2\geq 9$.

Now,

$$2(x^2+y^2+z^2)\geq 3(x+y+z)\Leftrightarrow \frac{2(x^2+y^2+z^2)}{3}\geq (x+y+z)$$

$$\Leftrightarrow 2\cdot\frac{2(x^2+y^2+z^2)}{3}\geq 2\cdot(x+y+z)$$

$$\Leftrightarrow \frac{4(x^2+y^2+z^2)}{3}\geq 2\cdot(x+y+z)$$

$$\Leftrightarrow x^2+y^2+z^2+\frac{(x^2+y^2+z^2)}{3}\geq 2\cdot(x+y+z)$$

$$\Leftrightarrow 3+\frac{x^2}{3}+3+\frac{y^2}{3}+3+\frac{z^2}{3}\geq 2(x+y+z)$$

$$\Leftrightarrow 3+\frac{x^2}{3}+3+\frac{y^2}{3}+3+\frac{z^2}{3}\geq 2(x+y+z)$$

$$\Leftrightarrow 2\sqrt{3\cdot\frac{x^2}{3}}+2\sqrt{3\cdot\frac{y^2}{3}}+2\sqrt{3\cdot\frac{z^2}{3}}\geq 2(x+y+z)$$

Equality holds if $3=\frac{x^2}{3}=\frac{y^2}{3}=\frac{z^2}{3}$, i.e. $x=y=z=3$, for which $x+y+z\neq xyz$.

Remark. The inequality can be improved: $x^2+y^2+z^2\geq\sqrt{3}(x+y+z)$

Solution. If one of the numbers is zero, then from $x+y+z=xyz$ all three numbers are zero and the equality trivially holds.

From AM-GM $x^2+y^2+z^2\geq 3\sqrt[3]{x^2y^2z^2}=3\frac{x+y+z}{\sqrt[3]{xyz}}\geq 3\frac{x+y+z}{\frac{x+y+z}{3}}=9$ (1).

From QM-AM $\frac{x^2+y^2+z^2}{3}\geq\left(\frac{x+y+z}{3}\right)^2$ (2).

Multiplying (1) and (2) we get $\frac{\left(x^2+y^2+z^2\right)^2}{3}\geq 9\frac{(x+y+z)^2}{9}=(x+y+z)^2$. By taking square root on both sides we deduce the stated inequality.

Equality holds only when $x=y=z=\sqrt{3}$ or $x=y=z=0$.

A6

Let a, b, c be positive real numbers. Prove that

$$\left((3a^2+1)^2+2\left(1+\frac{3}{b}\right)^2\right)\left((3b^2+1)^2+2\left(1+\frac{3}{c}\right)^2\right)\left((3c^2+1)^2+2\left(1+\frac{3}{a}\right)^2\right)\geq 48^3 .$$

When does equality hold?

Solution. Let x be a positive real number. By AM-GM we have $\frac{1+x+x+x}{4}\geq x^{\frac{3}{4}}$, or equivalently $1+3x\geq 4x^{\frac{3}{4}}$. Using this inequality we obtain:

$$(3a^2+1)^2\geq 16a^3 \text{ and } 2\left(1+\frac{3}{b}\right)^2\geq 32b^{-\frac{3}{2}} .$$

Moreover, by inequality of arithmetic and geometric means we have

$$f(a,b)=(3a^2+1)^2+2\left(1+\frac{3}{b}\right)^2\geq 16a^3+32b^{-\frac{3}{2}}=16\left(a^3+b^{-\frac{3}{2}}+b^{-\frac{3}{2}}\right)\geq 48\frac{a}{b} .$$

Therefore, we obtain

$$f(a,b)f(b,c)f(c,a)\geq 48\cdot\frac{a}{b}\cdot 48\cdot\frac{b}{c}\cdot 48\cdot\frac{c}{a}=48^3 .$$

Equality holds only when $a=b=c=1$.

A7

Let a,b,c be positive real numbers such that $a^2+b^2+c^2=48$. Prove

$$a^2\sqrt{2b^3+16}+b^2\sqrt{2c^3+16}+c^2\sqrt{2a^3+16}\le 24^2.$$

When does equality hold?

Solution. Observe that $2x^3+16=2(x^3+8)=2(x+2)(x^2-2x+4)$. From AM-GM:

$$\sqrt{2x^3+16}=\sqrt{(2x+4)(x^2-2x+4)}\le\frac{2x+4+x^2-2x+4}{2}=\frac{x^2+8}{2}\qquad(1).$$

By adding the inequality (1) obtained for $x=a$, $x=b$ and $x=c$ it suffices to prove:

$$a^2b^2+8a^2+b^2c^2+8b^2+c^2a^2+8c^2\le 2\cdot 24^2.$$

Since $a^2b^2+b^2c^2+c^2a^2\le\dfrac{(a^2+b^2+c^2)^2}{3}$, using $a^2+b^2+c^2=48$, we get the stated inequality.

Equality holds only when $a=b=c=4$.

A8

Let x, y and z be positive real numbers such that $xyz=1$. *Prove the inequality*

$$\frac{1}{x(ay+b)}+\frac{1}{y(az+b)}+\frac{1}{z(ax+b)}\geq 3, \text{ if:}$$

a) $a=0$ *and* $b=1$; *b)* $a=1$ *and* $b=0$; *c)* $a+b=1$ *for* $a,b>0$

When does the equality hold true?

Solution. a) The inequality reduces to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq 3$, which follows directly from the AM-GM inequality.

Equality holds only when $x=y=z=1$.

b) Here the inequality reduces to $\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\geq 3$, i.e. $x+y+z\geq 3$, which also follows from the AM-GM inequality.

Equality holds only when $x=y=z=1$.

c) Let *m*, *n* and *p* be such that $x=\frac{m}{n}$, $y=\frac{n}{p}$ и $z=\frac{p}{m}$. The inequality reduces to

$$\frac{np}{amn+bmp}+\frac{pm}{anp+bnm}+\frac{mn}{apm+bpn}\geq 3 \qquad (1).$$

By substituting $u=np$, $v=pm$ and $w=mn$, (1) becomes

$$\frac{u}{aw+bv}+\frac{v}{au+bw}+\frac{w}{av+bu}\geq 3.$$

The last inequality is equivalent to

$$\frac{u^2}{auw+buv}+\frac{v^2}{auv+bvw}+\frac{w^2}{avw+buw}\geq 3.$$

Cauchy-Schwarz Inequality implies

$$\frac{u^2}{auw+buv}+\frac{v^2}{auv+bvw}+\frac{w^2}{avw+buw}\geq\frac{(u+v+w)^2}{auw+buv+auv+bvw+avw+buw}=\frac{(u+v+w)^2}{uw+vu+wv}.$$

Thus, the problem simplifies to $(u+v+w)^2\geq 3(uw+vu+wv)$, which is equivalent to $(u-v)^2+(v-w)^2+(w-u)^2\geq 0$.

Equality holds only when $u=v=w$, that is only for $x=y=z=1$.

Remark. The problem can be reformulated:

Let a,b, x, y and z be nonnegative real numbers such that $xyz=1$ and $a+b=1$. Prove the inequality

$$\frac{1}{x(ay+b)}+\frac{1}{y(az+b)}+\frac{1}{z(ax+b)}\geq 3.$$

When does the equality hold true?

A9

Let n be a positive integer, and let $x_1,\dots,x_n,y_1,\dots,y_n$ be positive real numbers such that $x_1+\dots+x_n=y_1+\dots+y_n=1$. Show that

$$|x_1-y_1|+\dots|x_n-y_n|\le 2-\min_{1\le i\le n}\frac{x_i}{y_i}-\min_{1\le i\le n}\frac{y_i}{x_i}.$$

Solution. Up to reordering the real numbers x_i and y_i, we may assume that $\frac{x_1}{y_1}\le\dots\le\frac{x_n}{y_n}$. Let $A=\frac{x_1}{y_1}$ and $B=\frac{x_n}{y_n}$, and $S=|x_1-y_1|+\dots|x_n-y_n|$. Our aim is to prove that $S\le 2-A-\frac{1}{B}$.

First, note that we cannot have $A>1$, since that would imply $x_i>y_i$ for all $i\le n$, hence $x_1+\dots+x_n>y_1+\dots+y_n$. Similarly, we cannot have $B<1$, since that would imply $x_i<y_i$ for all $i\le n$, hence $x_1+\dots+x_n<y_1+\dots+y_n$.

If $n=1$, then $x_1=y_1=A=B=1$ and $S=0$, hence $S\le 2-A-\frac{1}{B}$. For $n\ge 2$ let $1\le k<n$ be some integer such that $\frac{x_k}{y_k}\le 1\le\frac{x_{k+1}}{y_{k+1}}$. We define the positive real numbers $X_1=x_1+\dots+x_k$, $X_2=x_{k+1}+\dots+x_n$, $Y_1=y_1+\dots+y_k$, $Y_2=y_{k+1}+\dots+y_n$. Note that $Y_1\ge X_1\ge AY_1$ and $Y_2\le X_2\le BY_2$. Thus, $A\le\frac{X_1}{Y_1}\le 1\le\frac{X_2}{Y_2}\le B$. In addition, $S=Y_1-X_1+X_2-Y_2$.

From $0<X_2,Y_1\le 1$, $0\le Y_1-X_1$ and $0\le X_2-Y_2$, follows

$$S=Y_1-X_1+X_2-Y_2=\frac{Y_1-X_1}{Y_1}+\frac{X_2-Y_2}{X_2}=2-\frac{X_1}{Y_1}-\frac{Y_2}{X_2}\le 2-A-\frac{1}{B}.$$

C1

Several (at least two) segments are drawn on a board. Select two of them, and let a and b be their lengths. Delete the selected segments and draw a segment of length $\frac{ab}{a+b}$. Continue this procedure until only one segment remains on the board. Prove:

a) the length of the last remaining segment does not depend on the order of the deletions.

b) for every positive integer n, the initial segments on the board can be chosen with distinct integer lengths, such that the last remaining segment has length n.

Solution. a) Observe that $\frac{1}{\frac{ab}{a+b}}=\frac{1}{a}+\frac{1}{b}$. Thus, if the lengths of the initial segments on the board were a_1, a_2, ..., a_n, and c is the length of the last remaining segment, then $\frac{1}{c}=\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}$, proving a).

b) From a) and the equation $\frac{1}{n}=\frac{1}{2n}+\frac{1}{3n}+\frac{1}{6n}$ it follows that if the lengths of the starting segments are $2n$, $3n$ and $6n$, then the length of the last remaining segment is n.

C2

In a country with n cities, all direct airlines are two-way. There are $r > 2014$ routes between pairs of different cities that include no more than one intermediate stop (the direction of each route matters). Find the least possible n and the least possible r for that value of n.

Solution. Denote by X_1, X_2, ... X_n the cities in the country and let X_i be connected to exactly m_i other cities by direct two-way airline. Then X_i is a final destination of m_i direct routes and an intermediate stop of $m_i(m_i - 1)$ non-direct routes. Thus $r = m_1^2 + ... + m_n^2$. As each m_i is at most $n-1$ and $13 \cdot 12^2 < 2014$, we deduce $n \geq 14$.

Consider $n = 14$. As each route appears in two opposite directions, r is even, so $r \geq 2016$. We can achieve $r = 2016$ by arranging the 14 cities uniformly on a circle connect (by direct two-way airlines) all of them, except the diametrically opposite pairs. This way, there are exactly $14 \cdot 12^2 = 2016$ routes.

C3

For a given positive integer n, two players A and B play the following game: Given is pile of a stones. The players take turn alternatively with A going first. On each turn the player is allowed to take one stone, a prime number of stones, or a multiple of n stones. The winner is the one who takes the last stone. Assuming perfect play, find the number of values for a, for which A cannot win.

Solution. Denote by k the sought number and let $\{a_1, a_2, \ldots, a_k\}$ be the corresponding values for a. We will call each a_i a losing number and every other positive integer a winning numbers. Clearly every multiple of n is a winning number.

Suppose there are two different losing numbers $a_i > a_j$, which are congruent modulo n. Then, on his first turn of play, the player A may remove $a_i - a_j$ stones (since $n | a_i - a_j$), leaving a pile with a_j stones for B. This is in contradiction with both a_i and a_j being losing numbers. Therefore there are at most $n-1$ losing numbers, i.e. $k \leq n-1$.

Suppose there exists an integer $r \in \{1, 2, \ldots, n-1\}$, such that $mn + r$ is a winning number for every $m \in \mathbb{N}_0$. Let us denote by u the greatest losing number (if $k > 0$) or 0 (if $k = 0$), and let $s = LCM(2, 3, \ldots, u+n+1)$. Note that all the numbers $s+2$, $s+3$, $\ldots$, $s+u+n+1$ are composite. Let $m' \in \mathbb{N}_0$, be such that $s+u+2 \leq m'n+r \leq s+u+n+1$. In order for $m'n+r$ to be a winning number, there must exist an integer p, which is either one, or prime, or a positive multiple of n, such that $m'n+r-p$ is a losing number or 0, and hence lesser than or equal to u. Since $s+2 \leq m'n+r-u \leq p \leq m'n+r \leq s+u+n+1$, p must be a composite, hence p is a multiple of n (say $p = qn$). But then $m'n+r-p = (m'-q)n+r$ must be a winning number, according to our assumption. This contradicts our assumption that all numbers $mn+r$, $m \in \mathbb{N}_0$ are winning.

Hence there are exactly $n-1$ losing numbers (one for each residue $r \in \{1, 2, \ldots, n-1\}$).

C4

Let $A = 1 \cdot 4 \cdot 7 \cdot \ldots \cdot 2014$ *be the product of the numbers less or equal to 2014 that give remainder 1 when divided by 3. Find the last non-zero digit of A.*

Solution. Grouping the elements of the product by ten we get:

$$(30k+1)(30k+4)(30k+7)(30k+10)(30k+13)(30k+16)$$
$$(30k+19)(30k+22)(30k+25)(30k+28)=$$
$$=(30k+1)(15k+2)(30k+7)(120k+40)(30k+13)(15k+8)$$
$$(30k+19)(15k+11)(120k+100)(15k+14)$$

(We divide all even numbers not divisible by five, by two and multiply all numbers divisible by five with four.)

We denote $P_k = (30k+1)(15k+2)(30k+7)(30k+13)(15k+8)(30k+19)(15k+11)(15k+14)$. For all the numbers not divisible by five, only the last digit affects the solution, since the power of two in the numbers divisible by five is greater than the power of five. Considering this, for even k, P_k ends with the same digit as $1 \cdot 2 \cdot 7 \cdot 3 \cdot 8 \cdot 9 \cdot 1 \cdot 4$, i.e. six and for odd k, P_k ends with the same digit as $1 \cdot 7 \cdot 7 \cdot 3 \cdot 3 \cdot 9 \cdot 6 \cdot 9$, i.e. six. Thus $P_0 P_1 \ldots P_{66}$ ends with six. If we remove one zero from the end of all numbers divisible with five, we get that the last nonzero digit of the given product is the same as the one from $6 \cdot 2011 \cdot 2014 \cdot 4 \cdot 10 \cdot 16 \cdot \ldots \cdot 796 \cdot 802$. Considering that $4 \cdot 6 \cdot 2 \cdot 8$ ends with four and removing one zero from every fifth number we get that the last nonzero digit is the same as in $4 \cdot 4^{26} \cdot 784 \cdot 796 \cdot 802 \cdot 1 \cdot 4 \cdot \ldots \cdot 76 \cdot 79$. Repeating the process we did for the starting sequence we conclude that the last nonzero number will be the same as in $2 \cdot 6 \cdot 6 \cdot 40 \cdot 100 \cdot 160 \cdot 220 \cdot 280 \cdot 61 \cdot 32 \cdot 67 \cdot 73 \cdot 38 \cdot 79$, which is two.

G1

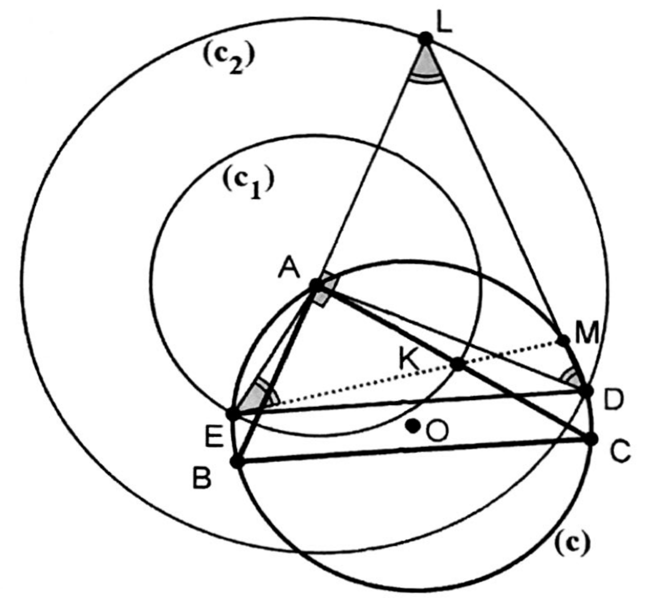
Let ABC *be a triangle with* $\measuredangle B = \measuredangle C = 40^{\circ}$. *The bisector of the* $\measuredangle B$ *meets* AC *at the point* D. *Prove that* $\overline{BD} + \overline{DA} = \overline{BC}$.

Solution. Since $\measuredangle BAC = 100^{\circ}$ and $\measuredangle BDC = 120^{\circ}$ we have $\overline{BD} < \overline{BC}$. Let E be the point on $\overline{BC}$ such that $\overline{BD} = \overline{BE}$. Then $\measuredangle DEC = 100^{\circ}$ and $\measuredangle EDC = 40^{\circ}$, hence $\overline{DE} = \overline{EC}$, and $\measuredangle BAC + \measuredangle DEB = 180^{\circ}$. So A, B, E and D are concyclic, implying $\overline{AD} = \overline{DE}$ (since $\measuredangle ABD = \measuredangle DBC = 20^{\circ}$), which completes the proof.

G2

Let ABC be an acute triangle with $\overline{AB} < \overline{AC} < \overline{BC}$ and $c(O,R)$ be its circumcircle. Denote with D and E be the points diametrically opposite to the points B and C, respectively. The circle $c_1(A,\overline{AE})$ intersects $\overline{AC}$ at point K, the circle $c_2(A,\overline{AD})$ intersects BA at point L (A lies between B and L). Prove that the lines EK and DL meet on the circle c.

Solution. Let M be the point of intersection of the line DL with the circle $c(O,R)$ (we choose $M \equiv D$ if LD is tangent to c and M to be the second intersecting point otherwise). It is sufficient to prove that the points E, K and M are collinear.



We have that $\measuredangle EAC = 90^\circ$ (since EC is diameter of the circle c). The triangle AEK is right-angled and isosceles ($\overline{AE}$ and $\overline{AK}$ are radii of the circle c_1). Therefore

$$\measuredangle AEK = \measuredangle AKE = 45^o.$$

Similarly, we obtain that $\measuredangle BAD = 90^\circ = \measuredangle DAL$. Since $\overline{AD} = \overline{AL}$ the triangle ADL is right-angled and isosceles, we have

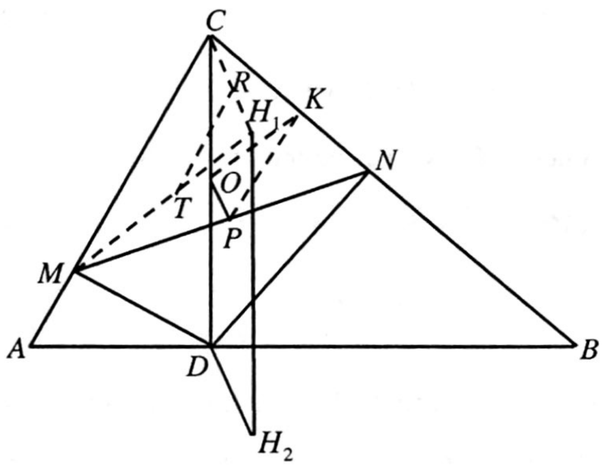
$$\measuredangle ADL = \measuredangle ALD = 45^o.$$

If M is between D and L, then $\measuredangle ADM = \measuredangle AEM$, because they are inscribed in the circle $c(O,R)$ and they correspond to the same arch $\overset{\frown}{AM}$. Hence $\measuredangle AEK = \measuredangle AEM = 45^o$ i.e. the points E,K,M are collinear.

If D is between M and L, then $\measuredangle ADM + \measuredangle AEM = 180^\circ$ as opposite angles in cyclic quadrilateral. Hence $\measuredangle AEK = \measuredangle AEM = 45^o$ i.e. the points E,K,M are collinear.

G3

Let $CD \perp AB$ ($D \in AB$), $DM \perp AC$ ($M \in AC$) and $DN \perp BC$ ($N \in BC$) for an acute triangle ABC with area S. If H_1 and H_2 are the orthocentres of the triangles MNC and MND respectively. Evaluate the area of the quadrilateral AH_1BH_2.



Solution1. Let O, P, K, R and T be the mid-points of the segments CD, MN, CN, CH_1 and MH_1, respectively. From ΔMNC we have that $\overline{PK} = \frac{1}{2}\overline{MC}$ and $PK \| MC$. Analogously, from ΔMH_1C we have that $\overline{TR} = \frac{1}{2}\overline{MC}$ and $TR \| MC$. Consequently, $\overline{PK} = \overline{TR}$ and $PK \| TR$. Also $OK \| DN$ (from ΔCDN) and since $DN \perp BC$ and $MH_1 \perp BC$, it follows that $TH_1 \| OK$. Since O is the circumcenter of ΔCMN, $OP \perp MN$. Thus, $CH_1 \perp MN$ implies $OP \| CH_1$. We conclude $\Delta TRH_1 \cong \Delta KPO$ (they have parallel sides and $\overline{TR} = \overline{PK}$), hence $\overline{RH_1} = \overline{PO}$, i.e. $\overline{CH_1} = 2\overline{PO}$ and $CH_1 \| PO$.

Analogously, $\overline{DH_2} = 2\overline{PO}$ and $DH_2 \| PO$. From $\overline{CH_1} = 2\overline{PO} = \overline{DH_2}$ and $CH_1 \| PO \| DH_2$ the quadrilateral CH_1H_2D is a parallelogram, thus $\overline{H_1H_2} = \overline{CD}$ and $H_1H_2 \| CD$. Therefore the area of the quadrilateral AH_1BH_2 is $\frac{\overline{AB} \cdot \overline{H_1H_2}}{2} = \frac{\overline{AB} \cdot \overline{CD}}{2} = S$.

Solution2. Since $MH_1 \| DN$ and $NH_1 \| DM$, $MDNH_1$ is a parallelogram. Similarly, $NH_2 \| CM$ and $MH_2 \| CN$ imply $MCNH_2$ is a parallelogram. Let P be the midpoint of the segment $\overline{MN}$. Then $\sigma_P(D) = H_1$ and $\sigma_P(C) = H_2$, thus $CD \| H_1H_2$ and $\overline{CD} = \overline{H_1H_2}$. From $CD \perp AB$ we deduce $A_{AH_1BH_2} = \frac{1}{2}\overline{AB} \cdot \overline{CD} = S$.

G4

Let ABC be a triangle such that $\overline{AB} \neq \overline{AC}$. Let M be a midpoint of $\overline{BC}$, H the orthocenter of ABC, O_1 the midpoint of $\overline{AH}$ and O_2 the circumcenter of BCH. Prove that O_1AMO_2 is a parallelogram.

Solution1. Let O_2' be the point such that O_1AMO_2' is a parallelogram. Note that $\overrightarrow{MO_2'} = \overrightarrow{AO_1} = \overrightarrow{O_1H}$. Therefore, $O_1HO_2'M$ is a parallelogram and $\overrightarrow{MO_1} = \overrightarrow{O_2'H}$.

Since M is the midpoint of $\overline{BC}$ and O_1 is the midpoint of $\overline{AH}$, it follows that $4\overrightarrow{MO_1} = \overrightarrow{BA} + \overrightarrow{BH} + \overrightarrow{CA} + \overrightarrow{CH} = 2(\overrightarrow{CA} + \overrightarrow{BH})$. Moreover, let B' be the midpoint of $\overline{BH}$. Then,

$$2\overrightarrow{O_2'B'} \cdot \overrightarrow{BH} = (\overrightarrow{O_2'H} + \overrightarrow{O_2'B}) \cdot \overrightarrow{BH} = (2\overrightarrow{O_2'H} + \overrightarrow{HB}) \cdot \overrightarrow{BH} =$$
$$= (2\overrightarrow{MO_1} + \overrightarrow{HB}) \cdot \overrightarrow{BH} = (\overrightarrow{CA} + \overrightarrow{BH} + \overrightarrow{HB}) \cdot \overrightarrow{BH} = \overrightarrow{CA} \cdot \overrightarrow{BH} = 0.$$

By $\vec{a} \cdot \vec{b}$ we denote the inner product of the vectors $\vec{a}$ and $\vec{b}$.

Therefore, O_2' lies on the perpendicular bisector of $\overline{BH}$. Since B and C play symmetric roles, O_2' also lies on the perpendicular bisector of $\overline{CH}$, hence O_2' is the circumcenter of ΔBCH and $O_2 = O_2'$.

Note: The condition $\overline{AB} \neq \overline{AC}$ just aims at ensuring that the parallelogram O_1ANO_2 is not degenerate, hence at helping students to focus on the "general" case.

Solution2. We use the following two well-known facts:

$\sigma_{BC}(H)$ lies on the circumcircle of ΔABC. (1)

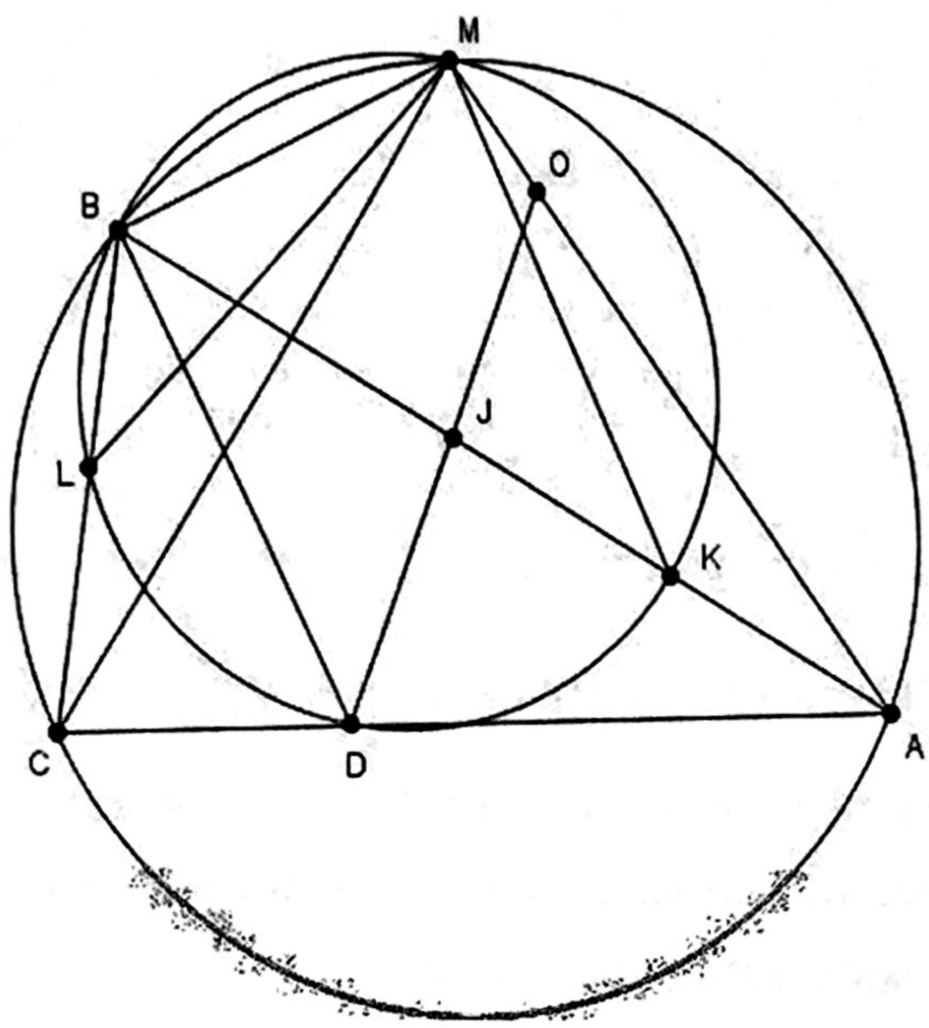
$\overrightarrow{AH} = -2\overrightarrow{MO}$, where O is the circumcenter of ΔABC. (2)

The statement "O_1AMO_2 is parallelogram" is equivalent to "$\sigma_{BC}(O_2) = O$". The later is true because the circumcircles of ΔABC and ΔBCH are symmetrical with respect to BC, from (1).

G5

Let ABC be a triangle with $\overline{AB} \neq \overline{BC}$, and let BD be the internal bisector of $\measuredangle ABC$ $(D \in AC)$. Denote the midpoint of the arc AC which contains point B by M. The circumcircle of the triangle BDM intersects the segment AB at point $K \neq B$, and let J be the reflection of A with respect to K. If $DJ \cap AM = \{O\}$, prove that the points J, B, M, O belong to the same circle.

Solution1.



Let the circumcircle of the triangle BDM intersect the line segment BC at point $L \neq B$. From $\measuredangle CBD = \measuredangle DBA$ we have $\overline{DL} = \overline{DK}$. Since $\measuredangle LCM = \measuredangle BCM = \measuredangle BAM = \measuredangle KAM$, $\overline{MC} = \overline{MA}$ and

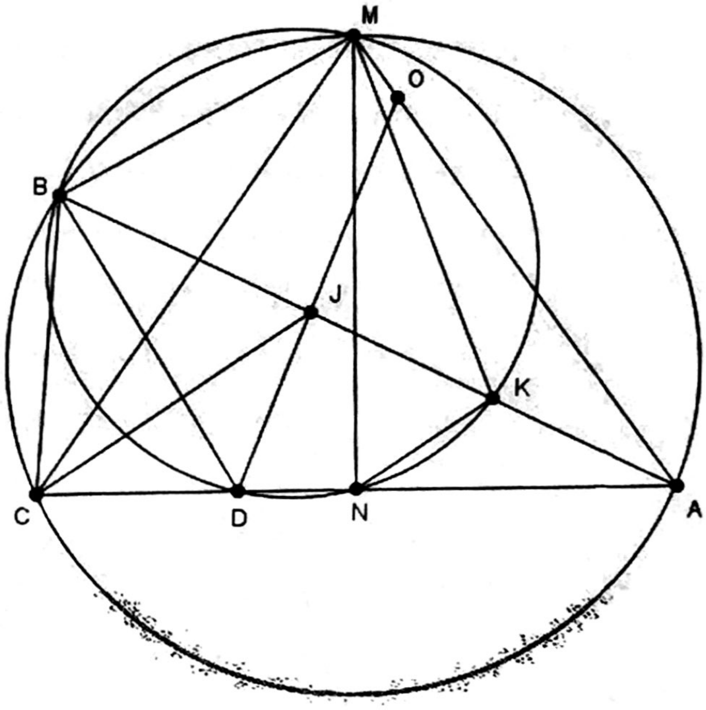
$$\measuredangle LMC = \measuredangle LMK - \measuredangle CMK = \measuredangle LBK - \measuredangle CMK = \measuredangle CBA - \measuredangle CMK = \measuredangle CMA - \measuredangle CMK = \measuredangle KMA,$$

it follows that triangles MLC and MKA are congruent, which implies $\overline{CL} = \overline{AK} = \overline{KJ}$. Furthermore, $\measuredangle CLD = 180° - \measuredangle BLD = \measuredangle DKB = \measuredangle DKJ$ and $\overline{DL} = \overline{DK}$, it follows that triangles DCL and DJK are congruent. Hence, $\measuredangle DCL = \measuredangle DJK = \measuredangle BJO$. Then

$$\measuredangle BJO + \measuredangle BMO = \measuredangle DCL + \measuredangle BMA = \measuredangle BCA + 180° - \measuredangle BCA = 180°$$

so the points J, B, M, O belong to the same circle, *q.e.d.*

Solution2.



Since $\overline{MC}=\overline{MA}$ and $\measuredangle CMA=\measuredangle CBA$, we have $\measuredangle ACM=\measuredangle CAM=90°-\frac{\measuredangle CBA}{2}$. It follows that $\measuredangle MBD=\measuredangle MBA+\measuredangle ABD=\measuredangle ACM+\measuredangle ABD=90°-\frac{\measuredangle CBA}{2}+\frac{\measuredangle CBA}{2}=90°$. Denote the midpoint of $\overline{AC}$ by N. Since $\measuredangle DNM=\measuredangle CNM=90°$, N belongs to the circumcircle of the triangle BDM. Since NK is the midline of the triangle ACJ and $NK \parallel CJ$, we have

$$\measuredangle BJC=\measuredangle BKN=180°-\measuredangle NDB=\measuredangle CDB.$$

Hence, the quadrilateral $CDJB$ is cyclic (this can also be obtained from the power of a point theorem, because $\overline{AN}\cdot\overline{AD}=\overline{AK}\cdot\overline{AB}$ implies $\overline{AC}\cdot\overline{AD}=\overline{AJ}\cdot\overline{AB}$), and

$$\measuredangle BJO=\measuredangle 180°-\measuredangle BJD=\measuredangle BCD=\measuredangle BCA=180°-\measuredangle BMA=180°-\measuredangle BMO,$$

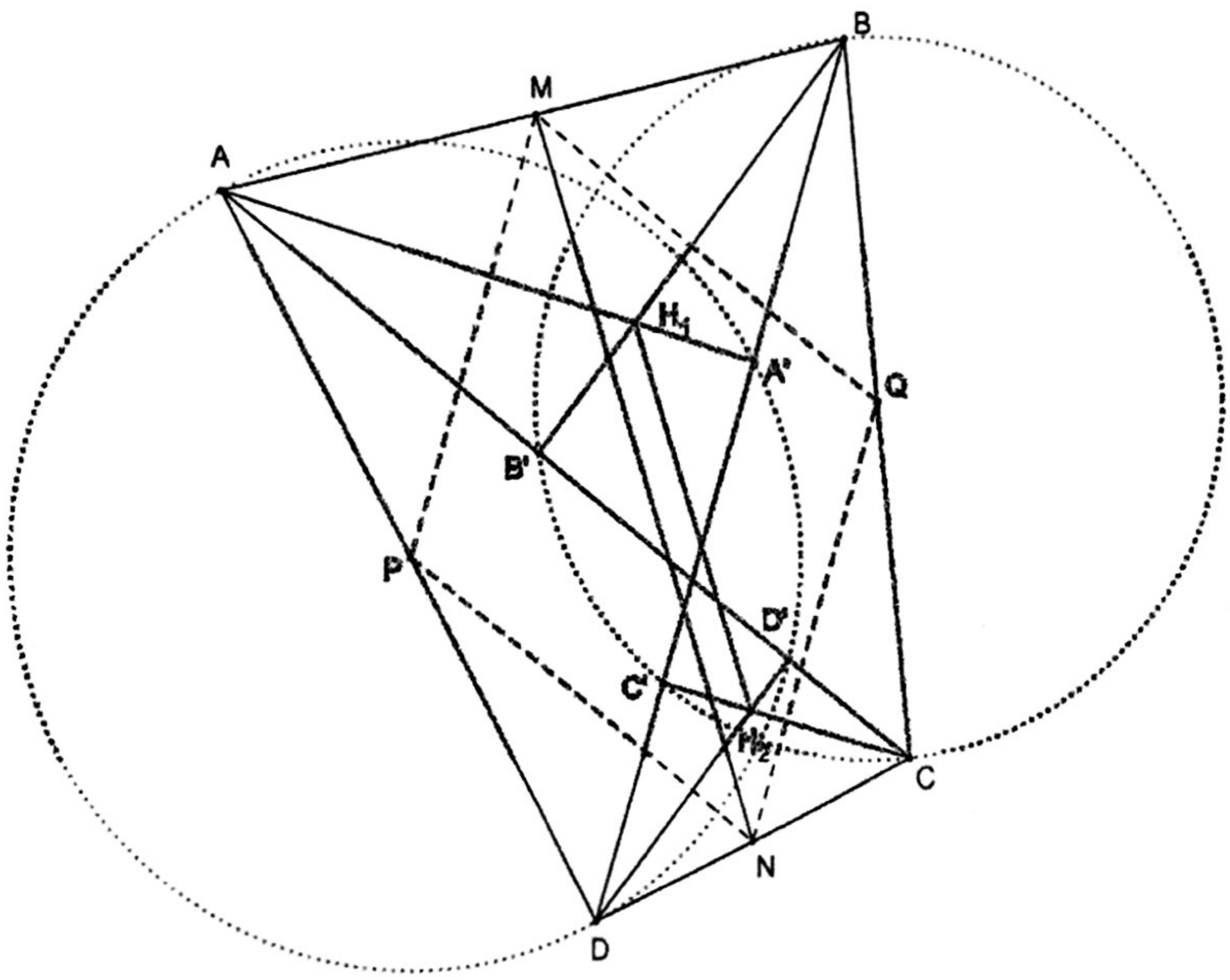
so the points J,B,M,O belong to the same circle, $q.e.d.$

Remark. If J is between A and K the solution can be easily adapted.

G6

Let $ABCD$ be a quadrilateral whose sides AB and CD are not parallel, and let O be the intersection of its diagonals. Denote with H_1 and H_2 the orthocenters of the triangles OAB and OCD, respectively. If M and N are the midpoints of the segments $\overline{AB}$ and $\overline{CD}$, respectively, prove that the lines MN and H_1H_2 are parallel if and only if $\overline{AC} = \overline{BD}$.

Solution.



Let A' and B' be the feet of the altitudes drawn from A and B respectively in the triangle AOB, and C' and D' are the feet of the altitudes drawn from C and D in the triangle COD. Obviously, A' and D' belong to the circle c_1 of diameter $\overline{AD}$, while B' and C' belong to the circle c_2 of diameter $\overline{BC}$.

It is easy to see that triangles H_1AB and $H_1B'A'$ are similar. It follows that $\overline{H_1A} \cdot \overline{H_1A'} = \overline{H_1B} \cdot \overline{H_1B'}$. (Alternatively, one could notice that the quadrilateral $ABA'B'$ is cyclic and obtain the previous relation by writing the power of H_1 with respect to its circumcircle.) It follows that H_1 has the same power with respect to circles c_1 and c_2. Thus, H_1 (and similarly, H_2) is on the radical axis of the two circles.

The radical axis being perpendicular to the line joining the centers of the two circles, one concludes that H_1H_2 is perpendicular to PQ, where P and Q are the midpoints of the sides $\overline{AD}$ and $\overline{BC}$, respectively. (P and Q are the centers of circles c_1 and c_2.)

The condition $H_1H_2 \parallel MN$ is equivalent to $MN \perp PQ$. As $MPNQ$ is a parallelogram, we conclude that $H_1H_2 \parallel MN \Leftrightarrow MN \perp PQ \Leftrightarrow MPNQ$ a rhombus $\Leftrightarrow \overline{MP} = \overline{MQ} \Leftrightarrow \overline{AC} = \overline{BD}$.

N1

Each letter of the word OHRID corresponds to a different digit belonging to the set {1,2,3,4,5}.

Decipher the equality $(O+H+R+I+D)^2:(O-H-R+I+D)=O^{H^{R^{I^D}}}$.

Solution. Since O, H, R, I and D are distinct numbers from $\{1,2,3,4,5\}$, we have $O+H+R+I+D=15$ and $O-H-R+I+D=O+H+R+I+D-2(H+R)<15$. From this $O^{H^{R^{I^D}}}=\frac{(O+H+R+I+D)^2}{O-H-R+I+D}=\frac{225}{15-2(H+R)}$, hence $O^{H^{R^{I^D}}}>15$ and divides 225, which is only possible for $O^{H^{R^{I^D}}}=25$ (must be a power of three or five). This implies that $O=5$, $H=2$ and $R=1$. It's easy to check that both $I=3$, $D=4$ and $I=4$, $D=3$ satisfy the stated equation.

N2

Find all triples (p,q,r) of distinct primes p, q and r such that

$$3p^4 - 5q^4 - 4r^2 = 26.$$

Solution. First notice that if both primes q and r differ from 3, then $q^2 \equiv r^2 \equiv 1 \pmod 3$, hence the left hand side of the given equation is congruent to zero modulo 3, which is impossible since 26 is not divisible by 3. Thus, $q=3$ or $r=3$. We consider two cases.

Case 1. $q=3$.

The equation reduces to $3p^4 - 4r^2 = 431$ (1).

If $p \neq 5$, by Fermat's little theorem, $p^4 \equiv 1 \pmod 5$, which yields $3 - 4r^2 \equiv 1 \pmod 5$, or equivalently, $r^2 + 2 \equiv 0 \pmod 5$. The last congruence is impossible in view of the fact that a residue of a square of a positive integer belongs to the set $\{0, 1, 4\}$. Therefore $p=5$ and $r=19$.

Case 2. $r=3$.

The equation becomes $3p^4 - 5q^4 = 62\,(2)$.

Obviously $p \neq 5$. Hence, Fermat's little theorem gives $p^4 \equiv 1 \pmod 5$. But then $5q^4 \equiv 1 \pmod 5$, which is impossible.

Hence, the only solution of the given equation is $p=5$, $q=3$, $r=19$.

N3

Find the integer solutions of the equation

$$x^2 = y^2(x+y^4+2y^2).$$

Solution. If $x=0$, then $y=0$ and conversely, if $y=0$, then $x=0$. It follows that $(x,y)=(0,0)$ is a solution of the problem. Assume $x\neq 0$ and $y\neq 0$ satisfy the equation. The equation can be transformed in the form $x^2-xy^2=y^6+2y^4$. Then $4x^2-4xy^2+y^4=4y^6+9y^4$ and consequently $\left(\frac{2x}{y^2}-1\right)^2=4y^2+9$ (1). Obviously $\frac{2x}{y^2}-1$ is integer. From (1), we get that the numbers $\frac{2x}{y^2}-1$, $2y$ and 3 are Pythagorean triplets. It follows that $\frac{2x}{y^2}-1=\pm 5$ and $2y=\pm 4$. Therefore, $x=3y^2$ or $x=-2y^2$ and $y=\pm 2$. Hence $(x,y)=(12,-2)$, $(x,y)=(12,2)$, $(x,y)=(-8,-2)$ and $(x,y)=(-8,2)$ are the possible solutions. By substituting them in the initial equation we verify that all the 4 pairs are solution. Thus, together with the couple $(x,y)=(0,0)$ the problem has 5 solutions.

N4

Prove there are no integers a and b satisfying the following conditions:

i) $16a-9b$ *is a prime number*

ii) ab *is a perfect square*

iii) $a+b$ *is a perfect square*

Solution. Suppose a and b be integers satisfying the given conditions. Let p be a prime number, n and m be integers. Then we can write the conditions as follows:

$$16a-9b=p \qquad (1)$$

$$ab=n^2 \qquad (2)$$

$$a+b=m^2 . \qquad (3)$$

Moreover, let $d=gdc(a,b)$ and $a=dx$, $b=dy$ for some relatively prime integers x and y.
Obviously $a\neq 0$ and $b\neq 0$, a and b are positive (by (2) and (3)).
From (2) follows that x and y are perfect squares, say $x=l^2$ and $y=s^2$.
From (1), $d \mid p$ and hence $d=p$ or $d=1$. If $d=p$, then $16x-9y=1$, and we obtain $x=9k+4$, $y=16k+7$ for some nonnegative integer k. But then $s^2=y\equiv 3(\bmod 4)$, which is a contradiction.
If $d=1$ then $16l^2-9s^2=p \Rightarrow (4l-3s)(4l+3s)=p \Rightarrow (4l+3s=p \wedge 4l-3s=1)$.
By adding the last two equations we get $8l=p+1$ and by subtracting them we get $6s=p-1$.
Therefore $p=24t+7$ for some integer t and $a=(3t+1)^2$ and $b=(4t+1)^2$ satisfy the conditions (1) and (2). By (3) we have $m^2=(3t+1)^2+(4t+1)^2=25t^2+14t+2$, or equivalently $25m^2=(25t+7)^2+1$.
Since the difference between two nonzero perfect square cannot be 1, we have a contradiction.
As a result there is no solution.

NS

Find all nonnegative integers x, y, z such that

$$2013^x + 2014^y = 2015^z.$$

Solution. Clearly, y > 0, and z > 0. If $x = 0$ and $y = 1$, then $z = 1$ and $(x, y, z) = (0,1,1)$ is a solution. If $x = 0$ and $y \geq 2$, then modulo 4 we have $1+0 \equiv (-1)^z$, hence z is even ($z = 2z_1$ for some integer z_1). Then $2^y 1007^y = (2015^{z_1} - 1)(2015^{z_1} + 1)$, and since $\gcd(1007, 2015^{z_1} + 1) = 1$ we obtain $2 \cdot 1007^y | 2015^{z_1} - 1$ and $2015^{z_1} + 1 | 2^{y-1}$. From this we get $2015^{z_1} + 1 \leq 2^{y-1} < 2 \cdot 1007^y \leq 2015^{z_1} - 1$, which is impossible.

Now for $x > 0$, modulo 3 we get $0+1 \equiv (-1)^z$, hence z must be even ($z = 2z_1$ for some integer z_1). Modulo 2014 we get $(-1)^x + 0 \equiv 1$, thus x must be even ($x = 2x_1$ for some integer x_1). We transform the equation to $2^y 1007^y = (2015^{z_1} - 2013^{x_1})(2015^{z_1} + 2013^{x_1})$ and since $\gcd(2015^{z_1} - 2013^{x_1}, 2015^{z_1} + 2013^{x_1}) = 2$, 1007^y divides $2015^{z_1} - 2013^{x_1}$ or $2015^{z_1} + 2013^{x_1}$ but not both. If $1007^y | 2015^{z_1} - 2013^{x_1}$, then $2015^{z_1} + 2013^{x_1} \leq 2^y < 1007^y \leq 2015^{z_1} - 2013^{x_1}$, which is impossible. Hence $1007 | 2015^{z_1} + 2013^{x_1}$, and from $2015^{z_1} + 2013^{x_1} \equiv 1 + (-1)^{x_1} \pmod{1007}$, x_1 is odd ($x = 2x_1 = 4x_2 + 2$ for some integer x_2).

Now modulo 5 we get $-1 + (-1)^y \equiv (-2)^{4x_2+2} + (-1)^y \equiv 0$, hence y must be even ($y = 2y_1$ for some integer y_1). Finally modulo 31, we have $(-2)^{4x_2+2} + (-1)^{2y_1} \equiv 0$ or $4^{2x_2+1} \equiv -1$. This is impossible since the reminders of the powers of 4 modulo 31 are 1, 2, 4, 8 and 16.

(Long Text)

N6

Vukasin, Dimitrije, Dusan, Stefan and Filip asked their professor to guess a three consecutive positive integer numbers after they had told him these (true) sentences:

Vukasin: "Sum of the digits of one of them is a prime number. Sum of the digits of some of the other two is an even perfect number (n is perfect if $\sigma(n)=2n$). Sum of the digits of the remaining number is equal to the number of its positive divisors."

Dimitrije:"Each of these three numbers has no more than two digits 1 in its decimal representation."

Dusan:"If we add 11 to one of them, we obtain a square of an integer."

Stefan:"Each of them has exactly one prime divisor less then 10."

Filip:"The 3 numbers are square-free."

Their professor gave the correct answer. Which numbers did he say?

Solution. Let the middle number be n, so the numbers are $n-1$, n and $n+1$. Since 4 does not divide any of them, $n \equiv 2 \pmod 4$. Furthermore, neither 3, 5 nor 7 divides n. Also $n+1+11 \equiv 2 \pmod 4$ cannot be a square. Then 3 must divide $n-1$ or $n+1$. If $n-1+11$ is a square, then $3|n+1$ which implies $3|n+10$ (a square), so $9|n+10$ hence $9|n+1$, which is impossible. Thus must be $n+11=m^2$.

Further, 7 does not divide $n-1$, nor $n+1$, because $1+11 \equiv 5 \pmod 7$ and $-1+11 \equiv 3 \pmod 7$ are quadratic nonresidues modulo 7. This implies $5|n-1$ or $5|n+1$. Again, since $n+11$ is a square, it is impossible $5|n-1$, hence $5|n+1$ which implies $3|n-1$. This yields $n \equiv 4 \pmod{10}$ hence $S(n+1)=S(n)+1=S(n-1)+2$ ($S(n)$ is sum of the digits of n). Since the three numbers are square-free, their numbers of positive divisors are powers of 2. Thus, we have two even sums of digits – they must be $S(n-1)$ and $S(n+1)$, so $S(n)$ is prime. From $3|n-1$, follows $S(n-1)$ is an even perfect number, and $S(n+1)=2^p$. Consequently $S(n)=2^p-1$ is a prime, so p is a prime number. One easily verifies $p \neq 2$, so p is odd implying $3|2^p-2$. Then

$\sigma(2^p-2)\geq(2^p-2)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{6}\right)=2(2^p-2)$. Since this number is perfect, $\frac{2^p-2}{6}$ must be one, i.e. $p=3$ and $S(n-1)=6$, $S(n)=7$ and $S(n+1)=8$.

Since 4 does not divide n, the 2-digit ending of n must be 14 or 34. But $n=34$ is impossible, since $n+11=45$ is not a square. Hence, $n=10^a+10^b+14$ with $a\geq b\geq 2$. If $a\neq b$, then n has three digits 1 in its decimal representation, which is impossible. Therefore $a=b$, and $n=2\cdot 10^a+14$. Now, $2\cdot 10^a+25=m^2$, hence $5|m$, say $m=5t$, and $(t-1)(t+1)=2^{a+1}5^{a-2}$.

Because $\gcd(t-1,t+1)=2$ there are three possibilities:

1) $t-1=2$, $t+1=2^a5^{a-2}$, which implies $a=2$, $t=3$;

2) $t-1=2^a$, $t+1=2\cdot 5^{a-2}$, so $2^a+2=2\cdot 5^{a-2}$, which implies $a=3$, $t=9$;

3) $t-1=2\cdot 5^{a-2}$, $t+1=2^a$, so $2\cdot 5^{a-2}+2=2^a$, which implies $a=2$, $t=3$, same as case 1).

From the only two possibilities $(n-1,n,n+1)=(213,214,215)$ and $(n-1,n,n+1)=(2013,2014,2015)$ the first one is not possible, because $S(215)=8$ and $\tau(215)=4$. By checking the conditions, we conclude that the latter is a solution, so the professor said the numbers: 2013, 2014, 2015.