CYP 7

17th Junior Balkan Mathematical Olympiad

Short-Listed Problems and Solutions

Antalya, Turkey 2013

The JBMO 2013 Problem Committee thanks the following countries for submitting problem proposals:

- Albania
- Bulgaria
- Bosnia and Herzegovina
- Cyprus
- Greece
- Indonesia
- Kazakhstan
- FYRO Macedonia
- Philippines
- Romania
- Serbia

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Problems

Algebra

HEL

A1. Find all ordered triples (x, y, z) of real numbers satisfying the following system of equations:

$$x^3 = \frac{z}{y} - 2\frac{y}{z}$$
$$y^3 = \frac{x}{z} - 2\frac{z}{x}$$
$$z^3 = \frac{y}{x} - 2\frac{x}{y}$$

ΙΝΔΟΝΗΣΙΑ

A2. Find the largest possible value of the expression $|\sqrt{x^2+4x+8} - \sqrt{x^2+8x+17}|$ where x is a real number.

SRB

A3. Show that

$$\left(a + 2b + \frac{2}{a+1}\right)\left(b + 2a + \frac{2}{b+1}\right) \geq 16$$

for all positive real numbers a, b satisfying $ab \geq 1$.

Combinatorics

C1. Find the largest number of distinct integers that can be chosen from the set $\{1, 2, \ldots, 2013\}$ so that the difference of no two of them is equal to 17.

C2. On a billiards table in the shape of a rectangle $ABCD$ with $AB = 2013$ and $AD = 1000$, a billiard ball is shot along the bisector of the angle $\angle BAD$. Assuming that the ball is reflected from the sides at the same angle it comes in, determine whether it will ever go to the corner B.

C3. All possible pairs of n apples are weighed and the results are given to us in an arbitrary order. Can we determine the weights of the apples if **a.** $n = 4$, **b.** $n = 5$, **c.** $n = 6$?

Geometry

G1. Let AB be a diameter of a circle ω with center O and OC be a radius of ω which is perpendicular to AB. Let M be a point on the line segment OC. Let N be the second point of intersection of the line AM with ω, and let P be the point of intersection of the lines tangent to ω at N and at B. Show that the points M, O, P, N are concyclic.

G2. ω_1 and ω_2 are two circles that are externally tangent to each other at the point M and internally tangent to a circle ω_3 at the points K and L, respectively. Let A and B be the two points where the common tangent line at M to ω_1 and ω_2 intersects ω_3. Show that if $\angle KAB = \angle LAB$ then the line segment AB is a diameter of ω_3.

G3. Let D be a point on the side BC of an acute triangle ABC such that $\angle BAD = \angle CAO$ where O is the center of the circumcircle ω of the triangle ABC. Let E be the second point of intersection of ω and the line AD. Let M, N, P be the midpoints of the line segments BE, OD, AC, respectively. Show that M, N, P are collinear.

G4. Let I be the incenter and AB the shortest side of a triangle ABC. The circle with center I and passing through C intersects the ray AB at the point P and the ray BA at the point Q. Let D be the point where the excircle of the triangle ABC belonging to angle A touches the side BC, and let E be the symmetric of the point C with respect to D. Show that the lines PE and CQ are perpendicular.

G5. A circle passing through the midpoint M of the side BC and the vertex A of a triangle ABC intersects the sides AB and AC for the second time at the points P and Q, respectively. Show that if $\angle BAC = 60°$ then

$$AP + AQ + PQ < AB + AC + \frac{1}{2} BC .$$

G6. Let P and Q be the midpoints of the sides BC and CD, respectively, of a rectangle $ABCD$. Let K and M be the points of intersection of the line PD with QB and QA, respectively, and let N be the point of intersection of the lines PA and QB.

Let X, Y, Z be the midpoints of the line segments AN, KN, AM, respectively. Let ℓ_1 be the line passing through X and perpendicular to MK, ℓ_2 be the line passing through Y and perpendicular to AM, ℓ_3 be the line passing through Z and perpendicular to KN. Show that ℓ_1, ℓ_2, ℓ_3 are concurrent.

Number Theory

#1B

N1. Find all positive integers n for which $1^3 + 2^3 + \cdots + 16^3 + 17^n$ is a perfect square.

KAZ ✓

N2. Find all ordered triples (x, y, z) of integers satisfying $20^x + 13^y = 2013^z$.

SRB

N3. Find all ordered pairs (a, b) of positive integers for which the numbers $\dfrac{a^3b - 1}{a + 1}$ and $\dfrac{b^3a + 1}{b - 1}$ are positive integers.

ROU

N4. A rectangle in the xy-plane is called *latticed* if all its vertices have integer coordinates.

a. Find a latticed rectangle with area 2013 whose sides are not parallel to the axes.

b. Show that if a latticed rectangle has area 2011, then its sides are parallel to the axes.

HEL

N5. Find all ordered triples (x, y, z) of positive integers satisfying the equation

$$\frac{1}{x^2} + \frac{y}{xz} + \frac{1}{z^2} = \frac{1}{2013}.$$

PHI

N6. Find all ordered triples (x, y, z) of integers satisfying the following system of equations:

$$x^2 - y^2 = z$$
$$3xy + (x - y)z = z^2$$

Solutions

A1. Find all ordered triples (x, y, z) of real numbers satisfying the following system of equations:

$$x^3 = \frac{z}{y} - 2\frac{y}{z}$$
$$y^3 = \frac{x}{z} - 2\frac{z}{x}$$
$$z^3 = \frac{y}{x} - 2\frac{x}{y}$$

Solution. We have

$$x^3yz = z^2 - 2y^2$$
$$y^3zx = x^2 - 2z^2$$
$$z^3xy = y^2 - 2x^2$$

with $xyz \neq 0$.

Adding these up we obtain $(x^2 + y^2 + z^2)(xyz + 1) = 0$. Hence $xyz = -1$. Now the system of equations becomes:

$$x^2 = 2y^2 - z^2$$
$$y^2 = 2z^2 - x^2$$
$$z^2 = 2x^2 - y^2$$

Then the first two equations give $x^2 = y^2 = z^2$. As $xyz = -1$, we conclude that $(x, y, z) = (1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1)$ and $(-1, -1, -1)$ are the only solutions.

A2. Find the largest possible value of the expression $|\sqrt{x^2+4x+8}-\sqrt{x^2+8x+17}|$ where x is a real number.

Solution. We observe that

$|\sqrt{x^2+4x+8}-\sqrt{x^2+8x+17}| = |\sqrt{(x-(-2))^2+(0-2)^2}-\sqrt{(x-(-4))^2+(0-1)^2)}|$

is the absolute difference of the distances from the point $P(x,0)$ in the xy-plane to the points $A(-2,2)$ and $B(-4,1)$.

By the Triangle Inequality, $|PA-PB| \le |AB|$ and the equality occurs exactly when P lies on the line passing through A and B, but not between them.

If P, A, B are collinear, then $(x-(-2))/(0-2) = ((-4)-(-2))/(1-2)$. This gives $x=-6$, and as $-6 < -4 < -2$,

$$|\sqrt{(-6)^2+4(-6)+8}-\sqrt{(-6)^2+8(-6)+17}| = |\sqrt{20}-\sqrt{5}| = \sqrt{5}$$

is the largest possible value of the expression.

A3. Show that

$$\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right)\geq 16$$

for all positive real numbers a, b satisfying $ab \geq 1$.

Solution 1. By the AM-GM Inequality we have:

$$\frac{a+1}{2}+\frac{2}{a+1}\geq 2$$

Therefore

$$a+2b+\frac{2}{a+1}\geq\frac{a+3}{2}+2b.$$

and, similarly,

$$b+2a+\frac{2}{b+1}\geq 2a+\frac{b+3}{2}.$$

On the other hand,

$$(a+4b+3)(b+4a+3)\geq(\sqrt{ab}+4\sqrt{ab}+3)^2\geq 64$$

by the Cauchy-Schwarz Inequality as $ab \geq 1$, and we are done.

Solution 2. Since $ab \geq 1$, we have $a+b\geq a+1/a\geq 2\sqrt{a\cdot(1/a)}=2$.

Then

$$\begin{aligned}
a+2b+\frac{2}{a+1} &= b+(a+b)+\frac{2}{a+1}\\
&\geq b+2+\frac{2}{a+1}\\
&= \frac{b+1}{2}+\frac{b+1}{2}+1+\frac{2}{a+1}\\
&\geq 4\sqrt[4]{\frac{(b+1)^2}{2(a+1)}}
\end{aligned}$$

by the AM-GM Inequality. Similarly,

$$b+2a+\frac{2}{b+1}\geq 4\sqrt[4]{\frac{(a+1)^2}{2(b+1)}}.$$

Now using these and applying the AM-GM Inequality another time we obtain:

$$\begin{aligned}
\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right) &\geq 16\sqrt[4]{\frac{(a+1)(b+1)}{4}}\\
&\geq 16\sqrt[4]{\frac{(2\sqrt{a})(2\sqrt{b})}{4}}\\
&= 16\sqrt[8]{ab}\\
&\geq 16
\end{aligned}$$

Solution 3. We have

$$\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right)=\left((a+b)+b+\frac{2}{a+1}\right)\left((a+b)+a+\frac{2}{b+1}\right)$$
$$\geq\left(a+b+\sqrt{ab}+\frac{2}{\sqrt{(a+1)(b+1)}}\right)^2$$

by the Cauchy-Schwarz Inequality.

On the other hand,

$$\frac{2}{\sqrt{(a+1)(b+1)}}\geq\frac{4}{a+b+2}$$

by the AM-GM Inequality and

$$a+b+\sqrt{ab}+\frac{2}{\sqrt{(a+1)(b+1)}}\geq a+b+1+\frac{4}{a+b+2}=\frac{(a+b+1)(a+b-2)}{a+b+2}+4\geq 4$$

as $a+b\geq 2\sqrt{ab}\geq 2$, finishing the proof.

C1. Find the largest number of distinct integers that can be chosen from the set $\{1, 2, \ldots, 2013\}$ so that the difference of no two of them is equal to 17.

Solution. Consider the sets $A_{mn} = \{34m + n - 34, 34m + n - 17\}$ for $1 \leq m \leq 59$ and $1 \leq n \leq 17$, and $B_k = \{2006 + k\}$ for $1 \leq k \leq 7$. As we cannot choose more than one number from each of these sets, we can choose at most $59 \cdot 17 + 7 = 1010$ numbers. On the other hand, choosing the smaller element of each of these sets gives exactly 1010 numbers satisfying the condition.

Comment. The original problem proposal asks the question with the numbers 55 and 5.

C2. On a billiards table in the shape of a rectangle $ABCD$ with $AB = 2013$ and $AD = 1000$, a billiard ball is shot along the bisector of the angle $\angle BAD$. Assuming that the ball is reflected from the sides at the same angle it comes in, determine whether it will ever go to the corner B.

Solution 1. The ball travels a horizontal distance of 1000 units between two bounces from the sides AB and CD as it always moves on a line making a 45° angle with the sides. Hence it is always at a distance of even number of units to the line AD when it hits AB or CD. Hence it can never hit AB at B.

Solution 2. Consider a rectangle $A'B'C'D'$ which is wider $1/2$ units on all sides, divide it into unit squares, and color them black and white alternatingly with the vertex A being the center of a black unit square. Then the ball always moves along the diagonals of the black unit squares. As B lies at the center of a white unit square, the ball never reaches B.

Solution 3. The vertical lines $x = 2013m$ and the horizontal lines $y = 1000n$, where m and n are integers, divide the xy-plane into rectangles congruent to the rectangle $ABCD$. Let $A(0,0)$, $B(2013,0)$, $C(2013,1000)$, $D(0,1000)$, and identify the other rectangles with $ABCD$ via reflections across these lines. Under this identification, the ball moves along the line $y = x$ and the coordinates of the points identified with B have the form $(2013k, 1000l)$ where k is an odd integer and l is an even one. Hence the ball never goes to B.

C3. All possible pairs of n apples are weighed and the results are given to us in an arbitrary order. Can we determine the weights of the apples if **a.** $n = 4$, **b.** $n = 5$, **c.** $n = 6$?

Solution. a. No. Four apples with weights $1, 5, 7, 9$ and with weights $2, 4, 6, 10$ both give the results $6, 8, 10, 12, 14, 16$ when weighed in pairs.

b. Yes. Let $a \le b \le c \le d \le e$ be the weights of the apples. As each apple is weighed 4 times, by adding all 10 pairwise weights and dividing the sum by 4, we obtain $a+b+c+d+e$. Subtracting the smallest and the largest pairwise weights $a+b$ and $d+e$ from this we obtain c. Subtracting c from the second largest pairwise weight $c+e$ we obtain e. Subtracting e from the largest pairwise weight $d+e$ we obtain d. a and b are similarly determined.

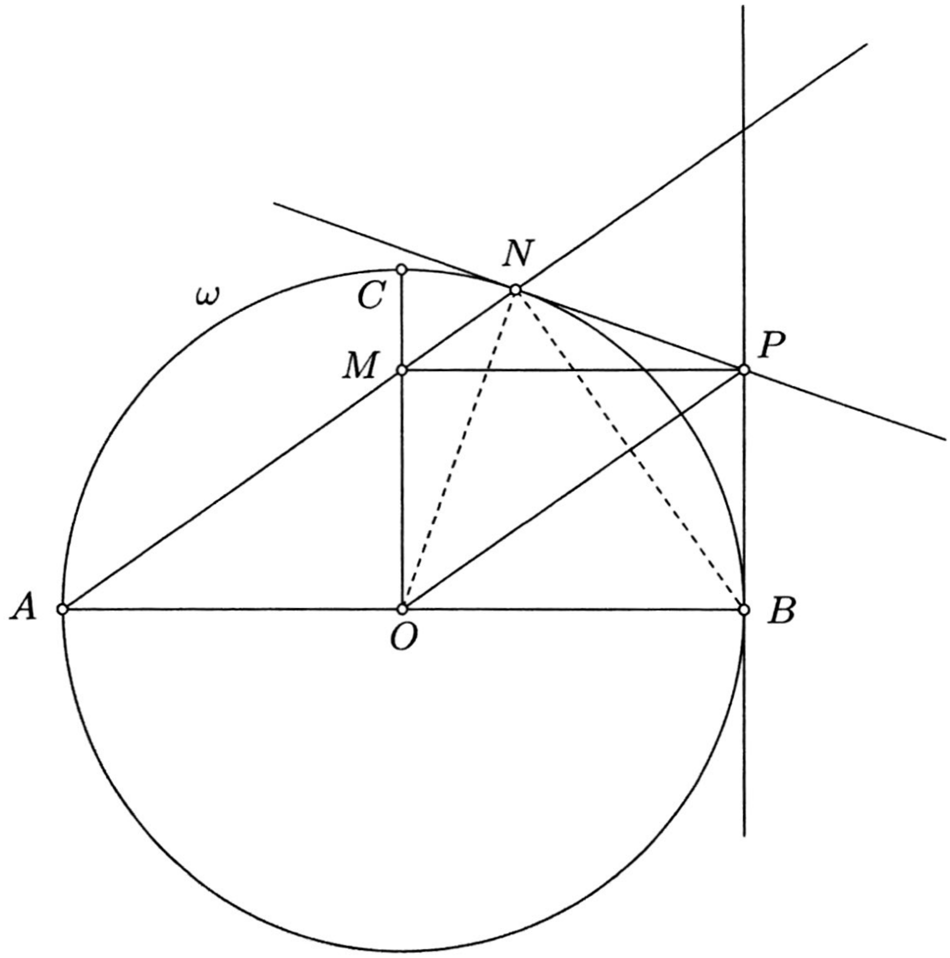
c. Yes. Let $a \le b \le c \le d \le e \le f$ be the weights of the apples. As each apple is weighed 5 times, by adding all 15 pairwise weights and dividing the sum by 5, we obtain $a+b+c+d+e+f$. Subtracting the smallest and the largest pairwise weights $a+b$ and $e+f$ from this we obtain $c+d$.

Subtracting the smallest and the second largest pairwise weights $a+b$ and $d+f$ from $a+b+c+d+e+f$ we obtain $c+e$. Similarly we obtain $b+d$. We use these to obtain $a+f$ and $b+e$.

Now $a+d$, $a+e$, $b+c$ are the three smallest among the remaining six pairwise weights. If we add these up, subtract the known weights $c+d$ and $b+e$ form the sum and divide the difference by 2, we obtain a. Then the rest follows.

G1. Let AB be a diameter of a circle ω with center O and OC be a radius of ω which is perpendicular to AB. Let M be a point on the line segment OC. Let N be the second point of intersection of the line AM with ω, and let P be the point of intersection of the lines tangent to ω at N and at B. Show that the points M, O, P, N are concyclic.

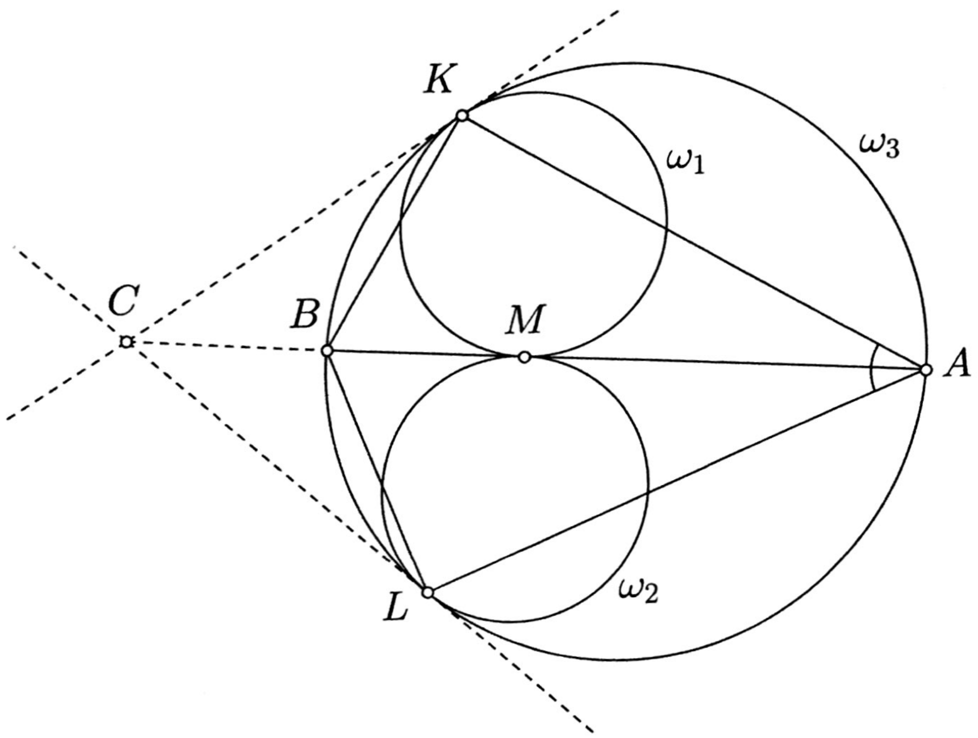
Solution. Since the lines PN and BP are tangent to ω, $NP = PB$ and OP is the bisector of $\angle NOB$. Therefore the lines OP and NB are perpendicular. Since $\angle ANB = 90°$, it follows that the lines AN and OP are parallel. As MO and PB are also parallel and $AO = OB$, the triangles AMO and OPB are congruent and $MO = PB$. Hence $MO = NP$. Therefore $MOPN$ is an isosceles trapezoid and therefore cyclic. Hence the points M, O, P, N are concyclic.



G2. ω_1 and ω_2 are two circles that are externally tangent to each other at the point M and internally tangent to a circle ω_3 at the points K and L, respectively. Let A and B be the two points where the common tangent line at M to ω_1 and ω_2 intersects ω_3. Show that if $\angle KAB = \angle LAB$ then the line segment AB is a diameter of ω_3.

Solution. Let C be the intersection point of the tangent lines to the circles ω_1 at K and ω_2 at L. Point C lies on the radical axis of circles ω_1 and ω_3, and also on the radical axis of the circles ω_2 and ω_3. Therefore C lies on the radical axis of the circles ω_1 and ω_2 too. Therefore the points A, B, C are collinear.

Since $\angle KAB = \angle LAB$, the chords KB and BL have the same length. As we also have $CK = CL$, the triangles KBC and LBC are congruent. In particular, $\angle KBA = \angle LBA$. Therefore, $\angle BKA = 180° - (\angle ABK + \angle BAK) = 180° - (\angle LBK + \angle LAK)/2 = 180° - 90° = 90°$, and AB is a diameter.



Comment. The original problem proposal gives $\angle KAB = \angle LAB = 15°$ and asks the measures of the angles of the quadrilateral $AKBL$.

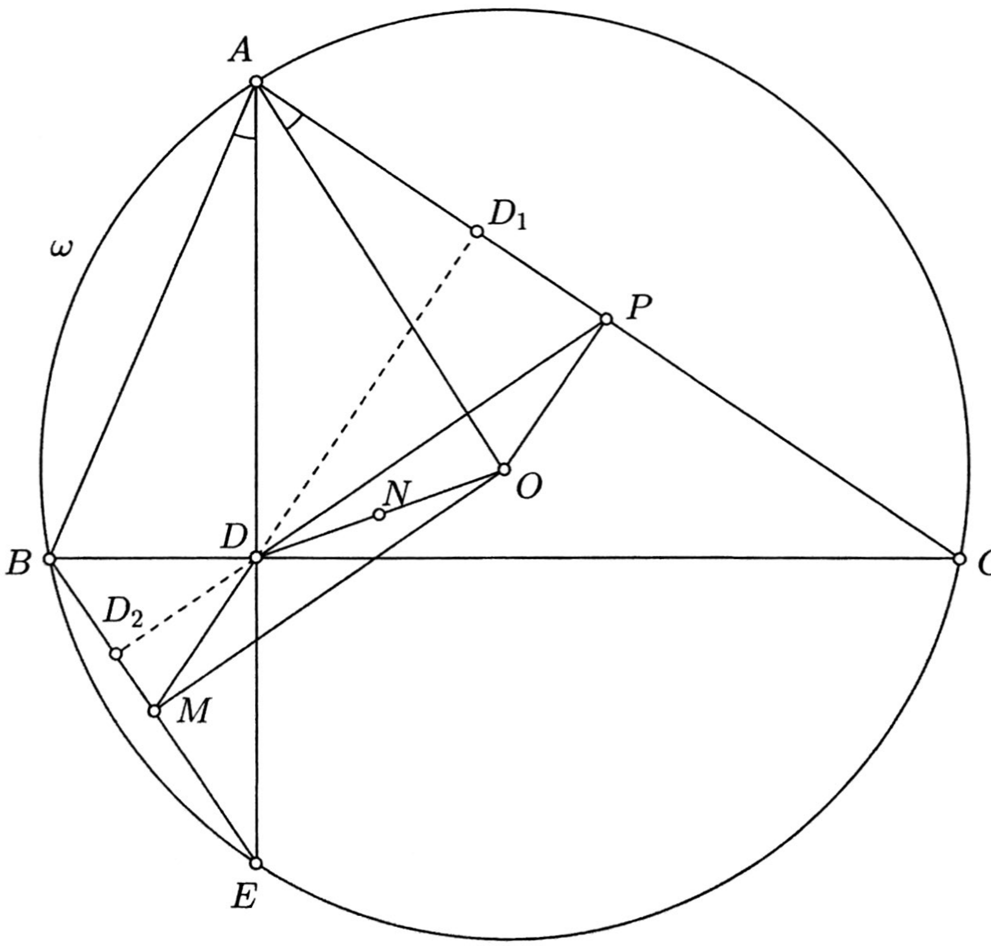
G3. Let D be a point on the side BC of an acute triangle ABC such that $\angle BAD = \angle CAO$ where O is the center of the circumcircle ω of the triangle ABC. Let E be the second point of intersection of ω and the line AD. Let M, N, P be the midpoints of the line segments BE, OD, AC, respectively. Show that M, N, P are collinear.

Solution. We will show that $MOPD$ is a parallelogram. From this it follows that M, N, P are collinear.

Since $\angle BAD = \angle CAO = 90° - \angle ABC$, D is the foot of the perpendicular from A to side BC. Since M is the midpoint of the line segment BE, we have $BM = ME = MD$ and hence $\angle MDE = \angle MED = \angle ACB$.

Let the line MD intersect the line AC at D_1. Since $\angle ADD_1 = \angle MDE = \angle ACD$, MD is perpendicular to AC. On the other hand, since O is the center of the circumcircle of triangle ABC and P is the midpoint of the side AC, OP is perpendicular to AC. Therefore MD and OP are parallel.

Similarly, since P is the midpoint of the side AC, we have $AP = PC = DP$ and hence $\angle PDC = \angle ACB$. Let the line PD intersect the line BE at D_2. Since $\angle BDD_2 = \angle PDC = \angle ACB = \angle BED$, we conclude that PD is perpendicular to BE. Since M is the midpoint of the line segment BE, OM is perpendicular to BE and hence OM and PD are parallel.

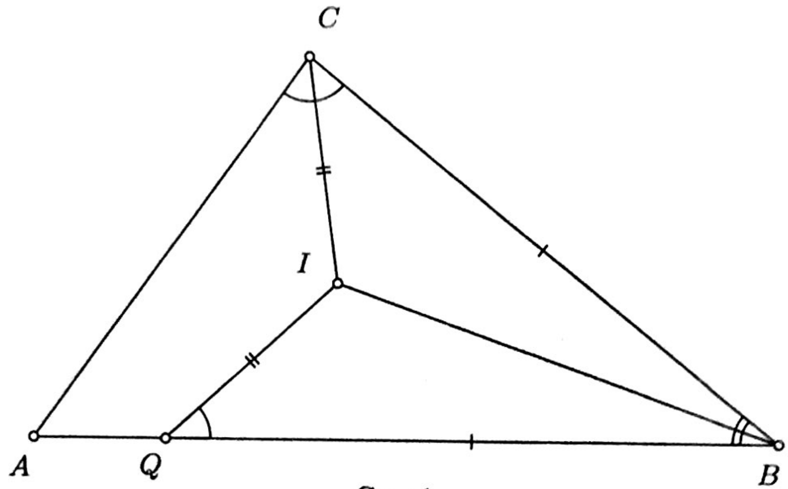


G4. Let I be the incenter and AB the shortest side of a triangle ABC. The circle with center I and passing through C intersects the ray AB at the point P and the ray BA at the point Q. Let D be the point where the excircle of the triangle ABC belonging to angle A touches the side BC, and let E be the symmetric of the point C with respect to D. Show that the lines PE and CQ are perpendicular.

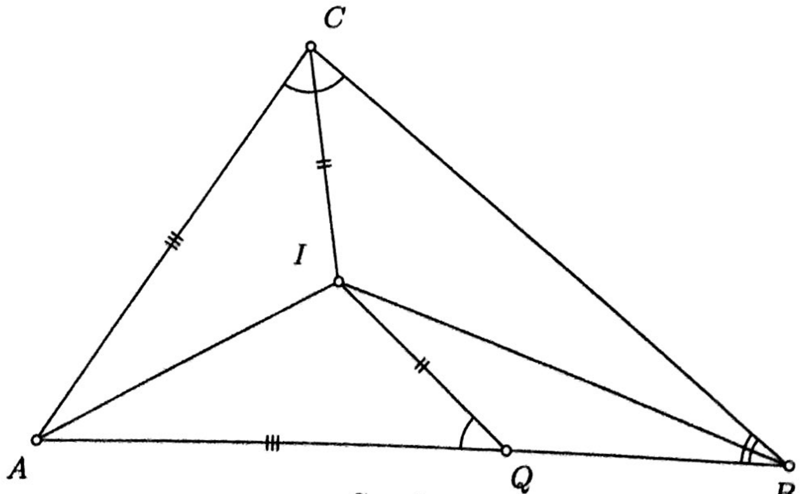
Solution. First we will show that points P and Q are not on the line segment AB.

Assume that Q is on the line segment AB. Since $CI = QI$ and $\angle IBQ = \angle IBC$, either the triangles CBI and QBI are congruent or $\angle ICB + \angle IQB = 180°$. In the first case, we have $BC = BQ$ which contradicts AB being the shortest side.

In the second case, we have $\angle IQA = \angle ICB = \angle ICA$ and the triangles IAC and IAQ are congruent. Hence this time we have $AC = AQ$, contradicting AB being the shortest side.



Case 1



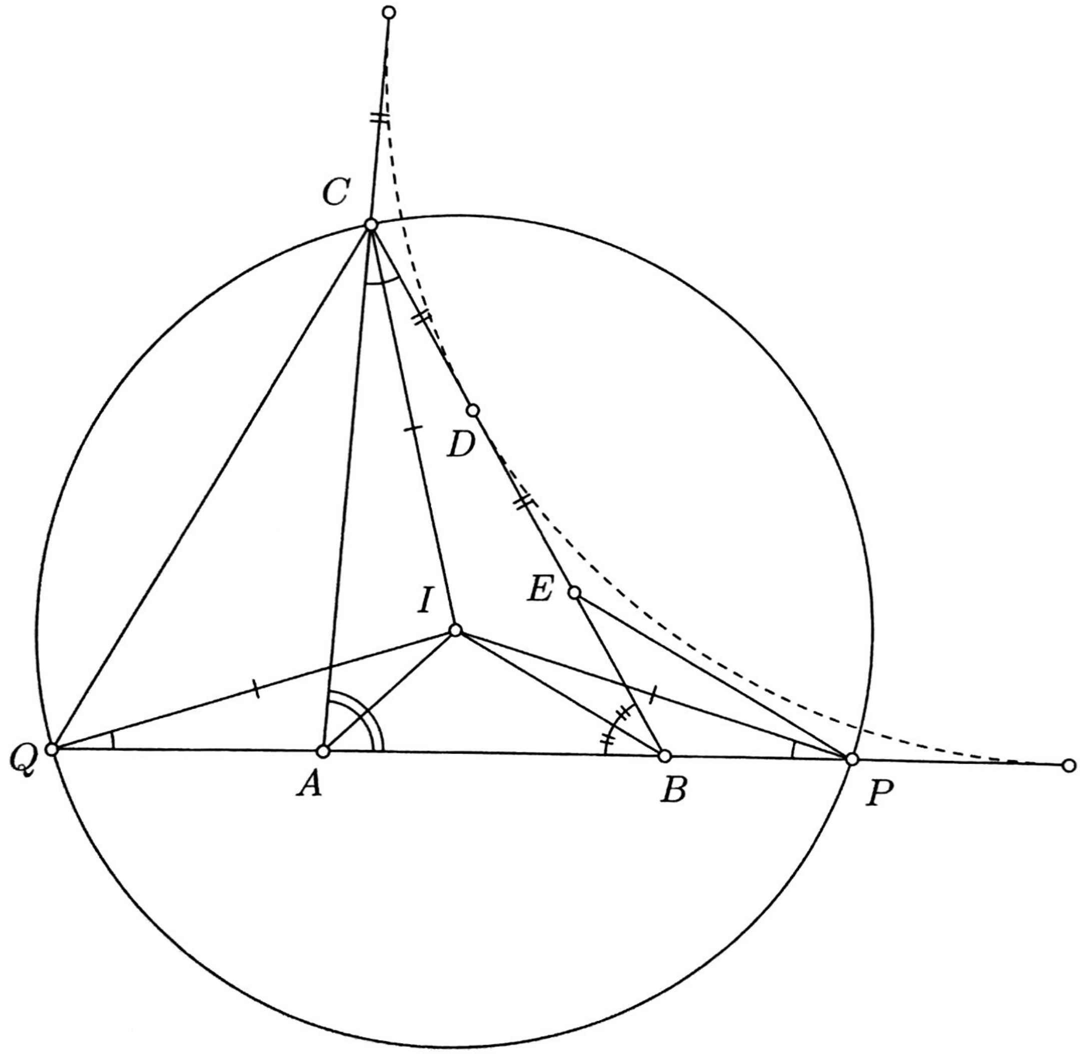
Case 2

Now we will show that the lines PE and CQ are perpendicular.

Since $\angle IQB \leq \angle IAB = (\angle CAB)/2 < 90°$ and $\angle ICB = (\angle ACB)/2 < 90°$, the triangles CBI and QBI are congruent. Hence $BC = BQ$ and $\angle CQP = \angle CQB = 90° - (\angle ABC)/2$. Similarly, we have $AC = AP$ and hence $BP = AC - AB$.

On the other hand, as $DE = CD$ and $CD + AC = u$, where u denotes the semiperimeter of the triangle ABC, we have $BE = BC - 2(u - AC) = AC - AB$. Therefore $BP = BE$ and $\angle QPE = (\angle ABC)/2$.

Hence, $\angle CQP + \angle QPE = 90°$.

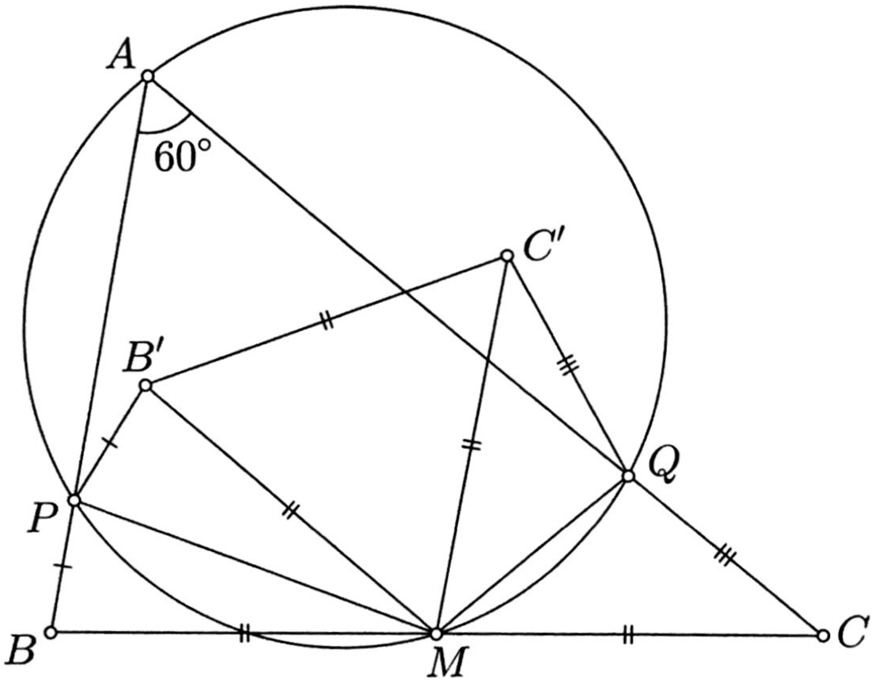


G5. A circle passing through the midpoint M of the side BC and the vertex A of a triangle ABC intersects the sides AB and AC for the second time at the points P and Q, respectively. Show that if $\angle BAC = 60°$ then

$$AP + AQ + PQ < AB + AC + \frac{1}{2}BC .$$

Solution. Since the quadrilateral $APMQ$ is cyclic, we have $\angle PMQ = 180° - \angle PAQ = 180° - \angle BAC = 120°$. Therefore $\angle PMB + \angle QMC = 180° - \angle PMQ = 60°$.

Let the point B' be the symmetric of the point B with respect to the line PM and the point C' be the symmetric of the point C with respect to the line QM. The triangles $B'MP$ and BMP are congruent and the triangles $C'MQ$ and CMQ are congruent. Hence $\angle B'MC' = \angle PMQ - \angle B'MP - \angle C'MQ = 120° - \angle BMP - \angle CMQ = 120° - 60° = 60°$. As we also have $B'M = BM = CM = C'M$, we conclude that the triangle $B'MC'$ is equilateral and $B'C' = BC/2$.



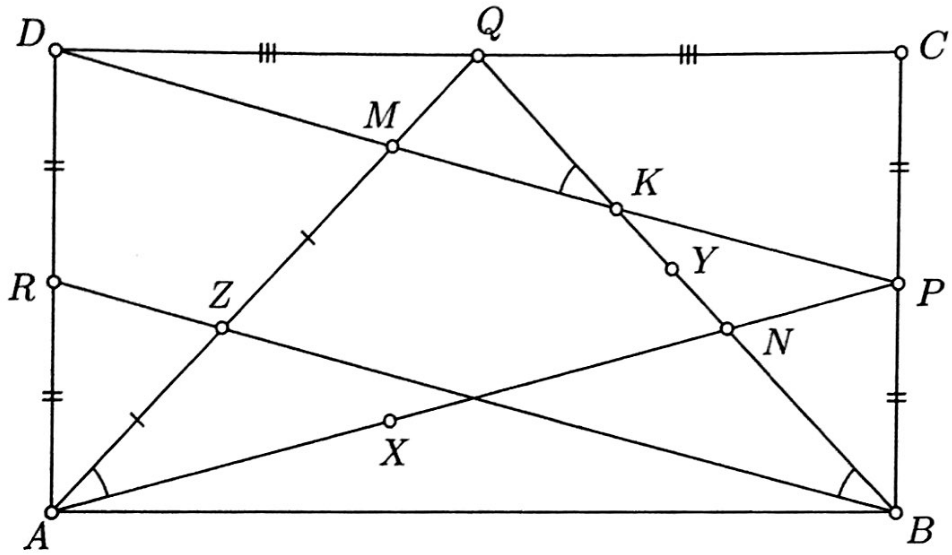
On the other hand, we have $PB' + B'C' + C'Q \geq PQ$ by the Triangle Inequality, and hence $PB + BC/2 + QC \geq PQ$. This gives the inequality $AB + BC/2 + AC \geq AP + PQ + AQ$.

We get an equality only when the points B' and C' lie on the line segment PQ. If this is the case, then $\angle PQC + \angle QPB = 2(\angle PQM + \angle QPM) = 120°$ and therefore $\angle APQ + \angle AQP = 240° \neq 120°$, a contradiction.

Let P and Q be the midpoints of the sides BC and CD, respectively, of a rectangle $ABCD$. Let K and M be the points of intersection of the line PD with QB and QA, respectively, and let N be the point of intersection of the lines PA and QB.

Let X, Y, Z be the midpoints of the line segments AN, KN, AM, respectively. Let ℓ_1 be the line passing through X and perpendicular to MK, ℓ_2 be the line passing through Y and perpendicular to AM, ℓ_3 be the line passing through Z and perpendicular to KN. Show that ℓ_1, ℓ_2, ℓ_3 are concurrent.

Solution. Let R be the midpoint of the side AD. Then the lines BR and PD are parallel. Since $\angle MAN = \angle QAP = \angle QBR = \angle QKM$, the points A, N, K, M are concyclic.



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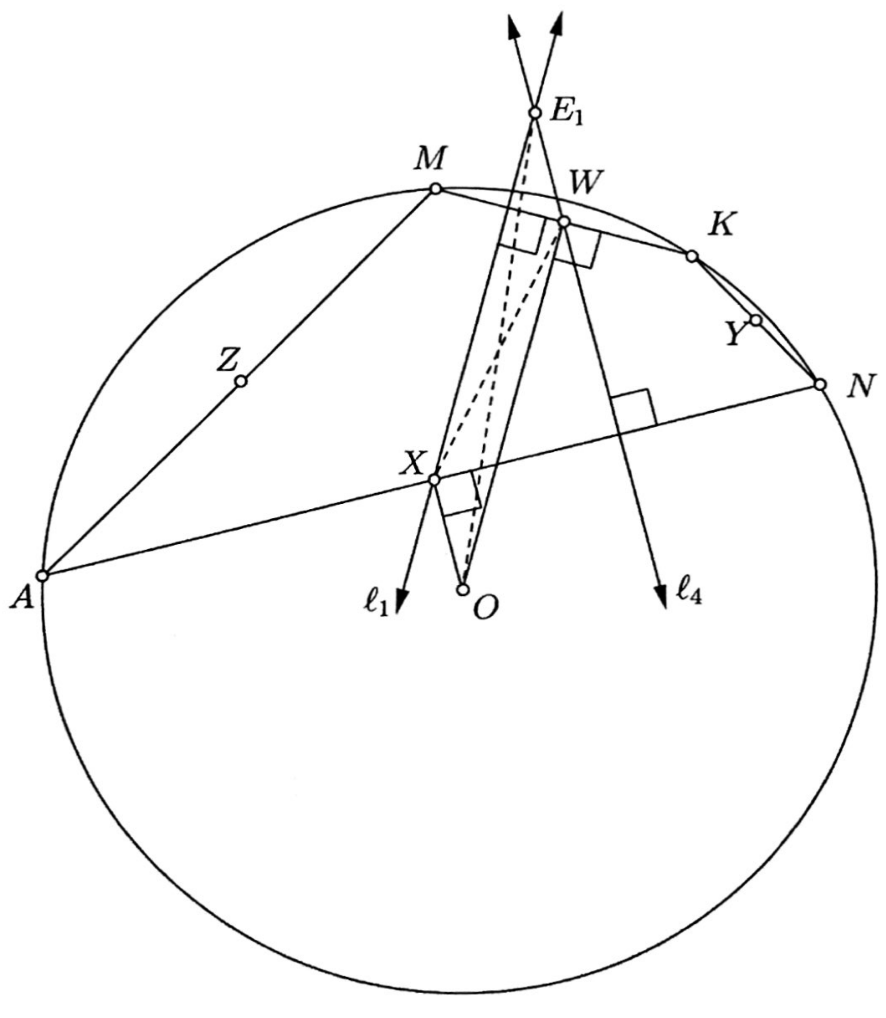
Let ℓ_4 be the line passing through the midpoint W of the line segment MK and perpendicular to the line AN. Let E_1 be the point of intersection of ℓ_1 and ℓ_4, and E_2 be the point of intersection of ℓ_2 and ℓ_3. We will show that the points E_1 and E_2 coincide.

Let O be the circumcenter of the cyclic quadrilateral $ANKM$. OW is perpendicular to the side MK and OX is perpendicular to the side AN. Hence OW is parallel to ℓ_1, OX is parallel to ℓ_4, and $XOWE_1$ is a parallelogram. Therefore the midpoints of the line segments OE_1 and WX coincide. Similarly, the midpoints of the line segments OE_2 and YZ coincide.

On the other hand, as X, Y, Z, W are midpoints of the sides of the quadrilateral $ANKM$, $XYWZ$ is a parallelogram and therefore the midpoints of the line segments WX and YZ coincide. Hence the midpoints of the line segments OE_1 and OE_2 coincide. In other words, E_1 and E_2 are the same point, and the lines ℓ_1, ℓ_2, ℓ_3 are concurrent.



Comment. The problem can be asked in the following form:

Let $ANKM$ be a cyclic quadrilateral and let X, Y, Z be the midpoints of the sides AN, KN, AM, respectively. Let ℓ_1 be the line passing through X and perpendicular to MK, ℓ_2 be the line passing through Y and perpendicular to AM, ℓ_3 be the line passing through Z and perpendicular to KN. Show that ℓ_1, ℓ_2, ℓ_3 are concurrent.

N1. Find all positive integers n for which $1^3 + 2^3 + \cdots + 16^3 + 17^n$ is a perfect square.

Solution. We have $1^3 + 2^3 + \cdots + 16^3 = (1 + 2 + \cdots + 16)^2 = 8^2 \cdot 17^2$. Hence, if $1^3 + 2^3 + \cdots + 16^3 + 17^n = m^2$ for a positive integer m, then $17 \mid m$. If $m = 17k$ for some positive integer k, then $17^{n-2} = (k + 8)(k - 8)$. As $(k + 8) - (k - 8) = 16$, this can only happen when $k + 8 = 17^{n-2}$ and $k - 8 = 1$. Hence $k = 9$, and $n = 3$ is the only solution.

N2. Find all ordered triples (x, y, z) of integers satisfying $20^x + 13^y = 2013^z$.

Solution. As $20 \cdot 13 = 2^2 \cdot 5 \cdot 13$ and $2013 = 3 \cdot 11 \cdot 61$ are relatively prime, x, y and z must be nonnegative.

Considering the equation modulo 3, we observe that x must be odd. Now considering the equation modulo 7, we obtain $(-1) + (-1)^y \equiv 4^z \pmod 7$, which is impossible as the right hand side can only be 1, 2 and 4 modulo 7.

There are no solutions.

N3. Find all ordered pairs (a, b) of positive integers for which the numbers $\frac{a^3b-1}{a+1}$ and $\frac{b^3a+1}{b-1}$ are positive integers.

Solution. As $a^3b - 1 = b(a^3 + 1) - (b + 1)$ and $a + 1 \mid a^3 + 1$, we have $a + 1 \mid b + 1$.

As $b^3a + 1 = a(b^3 - 1) + (a + 1)$ and $b - 1 \mid b^3 - 1$, we have $b - 1 \mid a + 1$.

So $b - 1 \mid b + 1$ and hence $b - 1 \mid 2$.

- If $b = 2$, then $a + 1 \mid b + 1 = 3$ gives $a = 2$. Hence $(a, b) = (2, 2)$ is the only solution in this case.
- If $b = 3$, then $a + 1 \mid b + 1 = 4$ gives $a = 1$ or $a = 3$. Hence $(a, b) = (1, 3)$ and $(3, 3)$ are the only solutions in this case.

To summarize, $(a, b) = (1, 3)$, $(2, 2)$ and $(3, 3)$ are the only solutions.

N4. A rectangle in the xy-plane is called *latticed* if all its vertices have integer coordinates.

a. Find a latticed rectangle with area 2013 whose sides are not parallel to the axes.

b. Show that if a latticed rectangle has area 2011, then its sides are parallel to the axes.

Solution. a. The rectangle $PQRS$ with $P(0,0)$, $Q(165,198)$, $R(159,203)$, $S(-6,5)$ has area 2013.

b. Suppose that the latticed rectangle $PQRS$ has area 2011 and its sides are not parallel to the axes. Without loss of generality we may assume that its vertices are $P(0,0)$, $Q(a,b)$, $S(c,d)$, $R(a+c,b+d)$ where a, b, c, d are integers and $abcd \neq 0$.

Then $(a^2+b^2)(c^2+d^2) = 2011^2$. As 2011 is a prime, $2011 \mid a^2+b^2$ or $2011 \mid c^2+d^2$. Assume that the first one is the case. Since $2011 \equiv 3 \pmod 4$, this can happen only if $a = 2011a_0$ and $b = 2011b_0$ for some integers a_0 and b_0. Now we have $(a_0^2+b_0^2)(c^2+d^2) = 1$. This means $c^2+d^2 = 1$ and hence $c=0$ or $d=0$, contradicting $cd \neq 0$.

Comment. The original problem proposal has the numbers 13 and 11 instead.

N5. Find all ordered triples (x, y, z) of positive integers satisfying the equation

$$\frac{1}{x^2} + \frac{y}{xz} + \frac{1}{z^2} = \frac{1}{2013}.$$

Solution. We have $x^2z^2 = 2013(x^2 + xyz + z^2)$. Let $d = \gcd(x, z)$ and $x = da$, $z = db$. Then $a^2b^2d^2 = 2013(a^2 + aby + b^2)$.

As $\gcd(a, b) = 1$, we also have $\gcd(a^2, a^2 + aby + b^2) = 1$ and $\gcd(b^2, a^2 + aby + b^2) = 1$. Therefore $a^2 \mid 2013$ and $b^2 \mid 2013$. But $2013 = 3 \cdot 11 \cdot 61$ is squarefree and therefore $a = 1 = b$.

Now we have $x = z = d$ and $d^2 = 2013(y + 2)$. Once again as 2013 is squarefree, we must have $y + 2 = 2013n^2$ where n is a positive integer.

Hence $(x, y, z) = (2013n, 2013n^2 - 2, 2013n)$ where n is a positive integer.

N6. Find all ordered triples (x, y, z) of integers satisfying the following system of equations:

$$x^2 - y^2 = z$$
$$3xy + (x - y)z = z^2$$

Solution. If $z = 0$, then $x = 0$ and $y = 0$, and $(x, y, z) = (0, 0, 0)$.

Let us assume that $z \neq 0$, and $x + y = a$ and $x - y = b$ where a and b are nonzero integers such that $z = ab$. Then $x = (a + b)/2$ and $y = (a - b)/2$, and the second equations gives $3a^2 - 3b^2 + 4ab^2 = 4a^2b^2$.

Hence

$$b^2 = \frac{3a^2}{4a^2 - 4a + 3}$$

and

$$3a^2 \geq 4a^2 - 4a + 3$$

which is satisfied only if $a = 1$, 2 or 3.

- If $a = 1$, then $b^2 = 1$. $(x, y, z) = (1, 0, 1)$ and $(0, 1, -1)$ are the only solutions in this case.
- If $a = 2$, then $b^2 = 12/11$. There are no solutions in this case.
- If $a = 3$, then $b^2 = 1$. $(x, y, z) = (1, 2, -3)$ and $(2, 1, 3)$ are the only solutions in this case.

To summarize, $(x, y, z) = (0, 0, 0)$, $(1, 0, 1)$, $(0, 1, -1)$, $(1, 2, -3)$ and $(2, 1, 3)$ are the only solutions.

Comments. 1. The original problem proposal asks for the solutions when $z = p$ is a prime number.

2. The problem can be asked with a single equation in the form:

$$3xy + (x - y)^2(x + y) = (x^2 - y^2)^2$$