Fifth team selection test
for Junior Balkan Mathematical Olympiad
Bucharest, May 25, 2013

Problem 18. Find all pairs of integers (x, y) satisfying the following condition: each of the numbers $x^3 + y$ and $x + y^3$ is divisible by $x^2 + y^2$.

Tournament of Towns

Solution. If $y = 0$, then $x^2 \mid x$, so $x \in \{-1, 0, 1\}$. Similarly, if $x = 0$, then $y \in \{-1, 0, 1\}$, so if $xy = 0$ we have five solutions:

$$(x, y) \in \{(0, 0), (1, 0), (0, 1), (-1, 0), (0, -1)\}.$$

Suppose that $xy \neq 0$ and let $d = (x, y)$, $x = du$, $y = dv$, $(u, v) = 1$. Since $x^2+y^2 \mid x^3+y$, we have $d^2 \mid d(d^2u^3 + v)$, so $d \mid v$. Likewise, from $x^2 + y^2 \mid x^3 + y$, we get that $d \mid u$, so $d \mid (u, v) = 1$, which means that x and y are relatively prime.

From $x^2 + y^2 \mid x^3 + y$ and $x^2 + y^2 \mid x(x^2 + y^2)$, we obtain $x^2 + y^2 \mid y(xy - 1)$. But $(x, y) = 1$ implies that $(x^2 + y^2, y) = 1$, so $x^2+y^2 \mid xy-1$, which leads to $x^2+y^2 \leq |xy - 1|$. It follows that

$$2|xy| \leq x^2 + y^2 \leq |xy - 1| \leq |xy| + 1,$$

so $|xy| \leq 1$. Since $xy \neq 0$, we have $|xy| = 1$, so $x, y \in \{-1, 1\}$ and all the possibilities $(x, y) \in \{(\pm 1, \pm 1)\}$ provide solutions.

Problem 19. Let $\mathcal{M}$ be the set of integer coordinate points situated on the line d of real numbers. We color the elements of $\mathcal{M}$ in black or white. Show that at least one of the following statements is true:

(a) there exists a finite subset $\mathcal{F} \subset \mathcal{M}$ and a point $M \in d$ so that the elements of the set $\mathcal{M} \setminus \mathcal{F}$ that are lying on one of the rays determined by M on d are all white, and the elements of $\mathcal{M} \setminus \mathcal{F}$ that are situated on the opposite ray are all black;

(b) there exists an infinite subset $\mathcal{S} \subset \mathcal{M}$ and a point $T \in d$ so that for each $A \in \mathcal{S}$, the reflection of A about T belongs to $\mathcal{S}$ and has the same color as A.

Marius Bocanu and Paul Musca

Solution. Consider the origin $O(0)$, the point $P(1/2)$ and the following sets:

- the set X of positive integers n with the property that $N(n)$ and $N_1(-n)$ have the same color;
- the set Y of positive integers n with the property that $N(n)$ and $N_2(1-n)$ have the same color.

Let's notice that $N_2(1-n)$ is the reflection of $N(n)$ about P.

If X is infinite, the statement (b) is true, taking $\mathcal{S} = X \cup (-X)$ and $T = O$. If X is finite, there is a positive integer a such that, for all $n \geq a$, one of the points $N(n)$ and $N_1(-n)$ is white and the other one is black.

If Y is infinite, then the statement (b) is true, taking $\mathcal{S} = Y \cup (1-Y)$ and $T = P$. Otherwise, there is a positive integer a such that, for all $n \geq b$, one of the points $N(n)$ and $N_1(1-n)$ is white and the other one is black.

If X and Y are finite, consider $n \geq m = \max\{a, b\}$ and suppose that $N(n)$ is white. It follows that $N_1(-n)$ is black. Since $n + 1 > m$, the points of coordinates $n + 1$ and

$1-(n+1)=-n$ have different colors, so $T(n+1)$ is white and, consequently, the point $R(-n-1)$ is black.

A simple induction proves that every point of coordinate $n \geq m$ is white and every point of coordinate $n \leq -m$ is black, so, considering the finite set $\mathcal{F}=\{n \in \mathbb{Z} \mid -m<n<m\}$ and $M=O(0)$, it follows that the statement (a) is true.

Remark. As a matter of fact, *exactly* one of the statements (a) or (b) is true. Indeed, if statement (b) is true, then there exists an infinite subset $\mathcal{T} \subset \mathcal{S} \subset \mathcal{M}$ with all points of the same color, for which the statement (b) is also true. That obviously contradicts statement (a).

Problem 20. Find the minimum and the maximum value of the expression

$$\sqrt{4-a^2}+\sqrt{4-b^2}+\sqrt{4-c^2},$$

where a, b, c are positive real numbers satisfying the condition $a^2+b^2+c^2=6$.

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Solution. Notice that $a, b, c \in [0,2]$. From Cauchy-Schwarz, we have

$$\left(\sqrt{4-a^2}+\sqrt{4-b^2}+\sqrt{4-c^2}\right)^2 \leq 3\left(4-a^2+4-b^2+4-c^2\right)=18,$$

so $\sqrt{4-a^2}+\sqrt{4-b^2}+\sqrt{4-c^2} \leq 3\sqrt{2}$, with equality for $|a|=|b|=|c|=\sqrt{2}$.

Obviously, for $x, y \geq 0$, we have $\sqrt{x}+\sqrt{y} \geq \sqrt{x+y}$, the equality occurring when $x=0$ or $y=0$. Suppose that $a \leq b \leq c$; then $6 \geq 3a^2$, so $a^2 \leq 2$.

It follows that

$$\sqrt{4-a^2}+\sqrt{4-b^2}+\sqrt{4-c^2} \geq \sqrt{4-a^2}+\sqrt{8-b^2-c^2}=\sqrt{4-a^2}+\sqrt{2+a^2},$$

and, since $0 \leq 4-c^2 \leq 4-b^2 \leq 4-a^2$, the equality holds for $4-c^2=0$, which means $c=2$.

It remains to find out the minimum value of the expression $\sqrt{4-x}+\sqrt{2+x}$,for $0 \leq x \leq 2$. We claim that

$$\sqrt{4-x}+\sqrt{2+x} \geq 2+\sqrt{2}.$$

Indeed, squaring the relation above we get $\sqrt{(4-x)(2+x)} \geq 2\sqrt{2}$, which is equivalent with $x(2-x) \geq 0$, obviously true. The inequality holds for $x=0$ or $x=2$. So the minimum value of the expression $\sqrt{4-a^2}+\sqrt{4-b^2}+\sqrt{4-c^2}$ is $2+\sqrt{2}$, which can be obtained for $\{a, b, c\}=\left\{0, \sqrt{2}, 2\right\}$.

Problem 21. Consider acute triangles ABC and BCD, with $\angle BAC=\angle BDC$, such that A and D are on opposite sides of line BC. Denote by E the foot of the perpendicular line to AC through B and by F the foot of the perpendicular line to BD through C. Let H_1 be the orthocenter of triangle ABC and H_2 be the orthocenter of BCD. Show that lines AD, EF and H_1H_2 are concurrent.

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Solution. Consider ω_1 and ω_2 the circumcircles of ABC and DBC, with radii R_1 and R_2. Since $\angle BH_1C=180^\circ-\angle BAC=180^\circ-\angle BDC$, the quadrilateral H_1BDC is cyclic, so $H_1 \in \omega_2$; similarly, we have $H_2 \in \omega_1$.

We have $BC = 2R_1 \sin BAC = 2R_2 \sin BDC$, so $R_1 = R_2$. Since $AH_1 = 2R_1 \cos BAC$ and $DH_2 = 2R_2 \cos BDC$, we get $AH_1 = DH_2$. But AH_1 and DH_2 are both perpendicular to BC, so they are parallel. It follows that AH_1DH_2 is a parallelogram, so AD and H_1H_2 meet in the midpoint P of segment $[H_1H_2]$.

Let M be the reflection of H_1 about E and N be the reflection of H_2 about F; we have then $M \in \omega_1$ and $N \in \omega_2$. Segment $[PE]$ is a midsegment of the triangle H_1H_2M, so $PE \parallel H_2M$. Similarly, $[PF]$ is a midsegment of the triangle H_2H_1N, so $PF \parallel H_1M$.

But $\angle CNH_1 = \angle CBH_1 = \angle CBM = \angle CH_2M$, so lines NH_1 and MH_2 are parallel, hence $P \in EF$, which concludes the proof.

Remark (Laurenţiu Ploscaru). The concurrence of lines AD, EF and H_1H_2 occurs even if $\angle BAC \neq \angle BDC$, regardless of the position of A and D relatively to line BC. Indeed, if G is the foot of the perpendicular line to AB through C and H is the foot of the perpendicular line to CD through B, the points E, F, G, H are all lying on the circle of diameter BC.

From Pascal's theorem for the hexagon $BHFCGE$, it follows that the intersection points of lines BH and CG, BE and CF, FH and GE are collinear. Now, look at the triangles GEH_1 and FHH_2; from Desargues' theorem it follows that lines GH, EF and H_1H_2 are concurrent.

With a similar argument, with Pascal's theorem for the hexagon $BFHCEG$ and Desargues' theorem for triangles AGE and DFH, we obtain that lines GH, EF and AD are concurrent.