Third team selection test
for Junior Balkan Mathematical Olympiad
Bucharest, April 27, 2013

Problem 10. Let n be a positive integer. Determine all positive integers p for which there exist positive integers $x_1 < x_2 < ... < x_n$ such that

$$\frac{1}{x_1} + \frac{2}{x_2} + ... + \frac{n}{x_n} = p.$$

Irish Mathematical Olympiad

Solution. Call *good* a number p for which there exist positive integers $x_1 < x_2 < ... < x_n$ such that $\frac{1}{x_1} + \frac{2}{x_2} + ... + \frac{n}{x_n} = p$.

Since $x_1, x_2, ..., x_n$ are integers and $x_1 < x_2 < ... < x_n$, we have $x_k \geq k$, so $\frac{k}{x_k} \leq 1$, for all $k = 1, 2, ..., n$. Then $\frac{1}{x_1} + \frac{2}{x_2} + ... + \frac{n}{x_n} \leq n$, so every *good* number, if there exists any, is between 1 and n.

Next, we will show that any integer $p \in \{1, 2, ..., n\}$ is *good*. Obviously, n is *good* (for $x_k = k$) and 1 is also *good* (take $x_k = kn$). For $2 \leq p \leq n - 1$, we write:

$$\sum_{k=1}^{n} \frac{k}{x_k} = \left(\frac{1}{x_1} + \frac{2}{x_2} + ... + \frac{p-1}{x_{p-1}}\right) + \left(\frac{p}{x_p} + ... + \frac{n}{x_n}\right),$$

so it is enough to choose $x_1, x_2, ..., x_n$ such that the first sum is equal to $p - 1$, and the second sum is equal to 1. We can do that by setting $x_k = k$, for $k = 1, 2, ..., p - 1$ and $x_k = k(n - p + 1)$, for $p \leq k \leq n$. Notice that $x_1 < x_2 < ... < x_n$ in all cases, so the *good* numbers are indeed $1, 2, ..., n$.

Problem 11. Find all positive integers x, y, z such that $7^x + 13^y = 8^z$.

Lucian Petrescu

Solution 1. Reducing modulo 3 and modulo 4, we must have that x and z are odd. Let $z = 2m + 1$ and $x = 2n + 1$, where m and n are natural number; we have then $7^{2n+1} + 13^y = 8^{2m+1}$.

a) If $n = 3s$, we have $7^{6s+1} + 13^y \equiv 7 \cdot 343^{2s} \equiv 7 \cdot 25^s \equiv 7 \cdot (-1)^s \pmod{13}$ and $8^{2m+1} = 8 \cdot 64^m \equiv 8 \cdot (-1)^m \pmod{13}$, so this case yields no solutions.

b) If $n = 3s + 2$, then $7^{6s+5} + 13^y \equiv 49 \cdot 343^{2s+1} \equiv 50 \cdot 25^s \equiv 11 \cdot (-1)^s \pmod{13}$ and $8^{2m+1} = 8 \cdot 64^m \equiv 8 \cdot (-1)^m \pmod{13}$; again, no solutions.

c) If $n = 3s + 1$, we have $7^{6s+3} + 13^y = 8^{2m+1}$, which leads to

$$\left(2^{2m+1} - 7^{2s+1}\right)\left(4^{2m+1} + 2^{2m+1} \cdot 7^{2s+1} + 49^{2s+1}\right) = 13^y.$$

Taking $a = 2^{2m+1}$, $b = 7^{2s+1}$, it is easy to show that $(a - b,\ a^2 + ab + b^2) = 1$, so $a - b = 1$ (and $a^2 + ab + b^2 = 13^y$). It follows that $2^{2m+1} = 7^{2s+1} + 1$, so $2^{2m+1} = 8 \cdot (7^{2s} - 7^{2s-1} + \ldots - 7 + 1)$.

If $m \geq 2$, we get $2^{2m-2} = 7^{2s} - 7^{2s-1} + \ldots - 7 + 1$, a contradiction, because the right hand sum is an odd number. For $m = 1$ we get $s = 0$ and the solution $x = 3$, $y = 2$, $z = 3$.

Solution 2. Notice that $7^x \equiv 8^z \pmod{13}$. According to Fermat's Little Theorem, we have $7^{12k} \equiv 1 \pmod{13}$ and $8^{12k} \equiv 1 \pmod{13}$, for each integer k. Studying the following table of residues modulo 13:

n	0	1	2	3	4	5	6	7	8	9	10	11
$7^n \pmod{13}$	**1**	7	10	**5**	9	11	**12**	6	3	**8**	4	2
$8^n \pmod{13}$	**1**	8	12	**5**	1	8	**12**	5	1	**8**	12	5

we deduce that $7^x \equiv 8^z \pmod{13}$ if and only if x and z are both divisible by 3. Set $x = 3t$, $z = 3u$; it follows that

$$13^y = 8^{3u} - 7^{3t} = \left(8^u - 7^t\right)\left(8^{2u} + 8^u 7^t + 7^{2t}\right).$$

Since $(8^u - 7^t, 8^{2u} + 8^u 7^t + 7^{2t}) \mid 3$, we must have $8^u - 7^t = 1$. But $7^t \pmod{16} \in \{1, 7\}$, so $8^u \pmod{16} \in \{2, 8\}$. This is possible only if $u = 1$, so $t = 1$, which leads to $x = 3$, $y = 2$, $z = 3$.

Problem 12. Let $ABCD$ be a cyclic quadrilateral and ω_1, ω_2 the incircles of triangles ABC and BCD. Show that the common external tangent line of ω_1 and ω_2, the other one than BC, is parallel with AD.

Ştefan Spătaru

Solution. Let I_1, I_2 be the incenters of ABC and BCD and t the other external tangent line of ω_1 and ω_2. The problem is trivial if lines AD and BC are parallel, so we'll suppose that AD is not parallel to BC. Denote by V the intersection point of AD and BC.

Since one of the exterior tangents of two circles is the reflection of the other one abou the line the passes through the centers, it follows that t, BC and I_1I_2 are concurrent in a point U.

We have $\angle BI_1C = 90° + \angle BAC = 90° + \angle BDC = \angle BI_2C$, so BCI_2I_1 is a cyclic quadrilateral. It follows that

$$\begin{aligned}\angle I_1UB &= \frac{1}{2}\left(m(\overset{\frown}{CI_2}) - m(\overset{\frown}{BI_1})\right) = \angle I_2BC - \angle I_1CB = \frac{1}{2}(\angle DBC - \angle ACB) = \\ &= \frac{1}{4}\left(m(\overset{\frown}{DC}) - m(\overset{\frown}{AB})\right) = \frac{1}{2}\angle AVB,\end{aligned}$$

hence $\angle(AD, BC) = 2\angle(I_1I_2, BC)$. This leads to $\angle(t, BC) = 2\angle(I_1I_2, BC) = \angle(AD, BC)$, so $t \parallel AD$.

Problem 13. Find all integers $n \geq 2$ with the property:
there is a permutation $(a_1, a_2, ..., a_n)$ of the set $\{1, 2, ..., n\}$ so that the numbers

$$a_1 + a_2 + ... + a_k, \; k = 1, 2, ...n,$$

have different remainders when divided by n.

Adapted by a problem of Emil Vasile

Solution. Let n be such a number. For every $k=\overline{1,n}$, set r_k the remainder obtained from the division of $s_k=a_1+a_2+...+a_k$ by n.

First, we prove that $a_1=n$. Arguing by contradiction, there would be $k\geq 2$ so that $a_k=n$. Since $s_k=s_{k-1}+n$, we have $n\mid s_k-s_{k-1}$, which means that $r_k=r_{k-1}$, false. We have then $a_1=n$ and $r_1=0$.

Since $s_n=\dfrac{n(n+1)}{2}$, if n would be an odd number, then $n\mid s_n$, so $r_n=0=r_1$, a contradiction. As a consequence, n must be even; we will show that every even number has the property required by the problem. Indeed, consider n even and, for $k=1,2,...,n$, define

$$a_k=\begin{cases} n+1-k, & k \text{ odd} \\ k-1, & k \text{ even}\end{cases}.$$

For each $k\in\left\{1,2,...,\frac{n}{2}\right\}$, we have

$$\begin{aligned} s_{2k-1} &= a_1+a_2+...+a_{2k-1}=k(n-1)+1; \\ s_{2k} &= a_1+a_2+...+a_{2k}=k(n+1), \end{aligned}$$

so

- $r_{2k-1}=k(n-1)+1-n\left\lfloor\dfrac{kn-(k-1)}{n}\right\rfloor=k(n-1)+1-n\cdot\left\lfloor k-\dfrac{k-1}{n}\right\rfloor=\begin{cases} 0 & ,k=1 \\ n-k+1, & k\geq 2\end{cases}.$
- $r_{2k}=k(n+1)-n\left\lfloor\dfrac{k(n+1)}{n}\right\rfloor=k(n+1)-n\cdot\left\lfloor k+\dfrac{k}{n}\right\rfloor=k(n+1)-nk=k.$

Obviously, if $a,b\in\{1,2,...,n-1\}$ have the same parity, then $r_a\neq r_b$. If there would exist $k,j\in\left\{1,2,...,\frac{n}{2}\right\}$ so that $r_{2k-1}=r_{2j}$, then $j=0$ or $j=n-k+1$, impossible.

Remark. The original problem, selected from the shortlist of the national round of romanian mathematical olympiad in 2006, stated like this:

Let n be a positive integer. A bijection $f:\{1,2,...,n\}\to\{1,2,...,n\}$ has the property $\mathcal{P}$ if

$$g(k)=f(1)+f(2)+...+f(k)-n\left\lfloor\frac{f(1)+f(2)+...+f(k)}{n}\right\rfloor,\ k=1,2,...,n$$

defines a bijection $g:\{1,2,...,n\}\to\{0,1,...,n-1\}$. Show that there exists functions with the property $\mathcal{P}$ if and only if n is even.