# Isogonal Conjugates

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**Abstract**

This is a short note on isogonality, intended to exhibit the uses of isogonality in mathematical olympiads.

## Contents

# § Notations

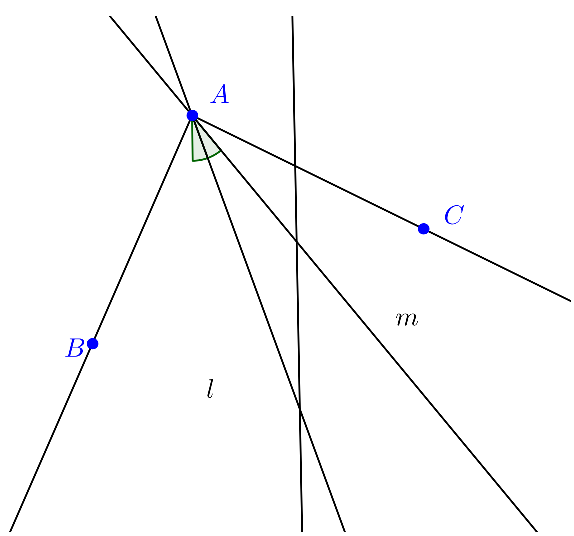
Here are some notations we'll use in this note.

1. The abbreviation 'wrt' stands for 'with respect to'.
2. $I, O, H, G$ stand for the incenter, circumcenter, orthocenter and the centroid of $\triangle ABC$ respectively.
3. $(XY), (XYZ)$ stand for the circle with $XY$ as diameter and the circle through non-collinear $X, Y, Z$ respectively.
4. $P^*$ stands for the isogonal conjugate of a point $P$.
5. $P_{\infty\|MN}$ stands for the point at infinity on the line $\overline{MN}$.
6. $\stackrel{P}{=}$ stands for equality of cross-ratios due to perspectivity at $P$.

# § Isogonal Lines

**Definition**

Two lines meeting at a point $A$ are said to be isogonal with respect to an angle $\measuredangle BAC$ if they can be obtained by a reflection over the angle bisector of $\measuredangle BAC$.




(1)

Also if one line (not necessarily through $A$) can be obtained from the other via a reflection about the angle bisector and a homothety, then they are called antiparallel with respect to the angle.

- ## Properties

In the following we let $l$ and $m$ be isogonal lines with respect to an angle $\measuredangle BAC$.

1. Let $P$ be a point on $l$ and the feet of perpendiculars from $P$ to $AB, AC$ be $P_c, P_b$. Then $P_bP_c \perp m$.

   *Proof.* Exercise. □

2. (Symmedians Lemma) Let $M$ be the midpoint of $\overline{BC}$ and let the tangents to the circumcircle meet at $X$. $AX \cap (ABC) = K$ and $AX \cap BC = J$. Then the following hold:

   – $AK$ is a symmedian in $\triangle ABC$.
   – $\frac{BJ}{JC} = (\frac{AB}{AC})^2$.
   – $BCAK$ is a harmonic quadrilateral.
   – $ABK \sim AMC$.
   – $(AO)$ and $(BOC)$ meet on the midpoint of $AK$.
   – $BC$ is a symmedian in both $BAK$ and $CAK$.
   – $BC$ is one angle bisector of $\measuredangle AMK$ and $MX$ is the other one.
   – Tangents to $(ABC)$ at $A$ and $K$ meet on $BC$.

   *Proof.* Exercise. One of the proofs of the first fact is in the examples. Try finding atleast 3 others. □

3. Let the cevians $AD_1$ and $AD_2$ be isogonal wrt $BAC$. Then the circumcircles of $AD_1D_2$ and $ABC$ are tangent together.

   *Proof.* Exercise. □

4. (Isogonal Line Lemma): Let $P$ and $Q$ be points on $l, m$ respectively. $BP$ intersects $CQ$ at $L_1$ and $BQ$ intersects $CP$ at $L_2$ . Then $AL_1$ and $AL_2$ are isogonals wrt $\measuredangle BAC$.

   *Proof.* Exercise. This also appears in St. Petersburg Mathematical Olympiad. □

5. Let $P$ be a point on the perpendicular bisector of $BC$ and let $P'$ be its inverse in the circumcircle. Then $AP$ and $AP'$ are isogonal wrt $\measuredangle BAC$.

   *Proof.* Follows from the Symmedians lemma. □

# § Isogonal Conjugates

**Definition**

Two points $P$ and $P^*$ are called isogonal conjugates if $\overline{AP}$ and $\overline{AP^*}$ are isogonal wrt $\measuredangle A$, $\overline{BP}$ and $\overline{BP^*}$ are isogonal wrt $\measuredangle B$ and $\overline{CP}$ and $\overline{CP^*}$ are isogonal wrt $\measuredangle C$.

*Proof.* Usually proved using trig Ceva, but see the following for a synthetic proof. □

## • Properties

1. Let $P$'s reflections in $BC, CA, AB$ be $P_a, P_b, P_c$ respectively. Then the circumcenter of the triangle $\triangle P_aP_bP_c$ is the isogonal conjugate of $P$, henceforth denoted by $P^*$.

   *Proof.* Let $Q$ be the required circumcenter. Then from $AQP_b \cong AQP_c$, $\measuredangle QAP_b = \measuredangle QAP_c$, from which $AQ$ and $AP$ are isogonal wrt $\measuredangle BAC$. From this and other similar relations, $P$ and $Q$ are isogonal conjugates, as required. □

2. If $D_1$ and $D_2$ are as in Property 3 of section on properties of isogonal lines, then $\frac{BD_1 \cdot BD_2}{CD_1 \cdot CD_2} = \frac{AB^2}{AC^2}$.

   *Proof.* Exercise □

3. The pedal triangles of $P$ and its isogonal conjugate have the same circumcircle with the circumcenter being the midpoint of $PP^*$.

   *Proof.* Homothety. □

4. $AP \cap BC = D$ and $AP^* \cap (ABC) = D'$. Then $AD.AD' = AB.AC$.

   *Proof.* Inversion □

5. $\measuredangle BPC + \measuredangle BP^*C = \measuredangle A$

   *Proof.* Angle chasing □

6. The insimilicenter and the exsimilicenter of $(BPC)$ and $(BP^*C)$ lie on $(ABC)$.

   *Proof.* Exercise. □

7. If the circumcenters of $(BPC)$ and $(BP^*C)$ are $O_1$ and $O_2$ then $AO_1$ and $AO_2$ are isogonal with respect to $\measuredangle BAC$.

*Proof.* Follows from the previous fact and Property 5 in the section on properties of isogonal lines. □

8. Let $\mathcal{E}$ be a conic with foci $F_1$ and $F_2$, with the tangents $t_1$ and $t_2$ from a point $X$ meeting it at $X_1$ and $X_2$. Then the following hold true:

   – $XX_1$ and $XX_2$ are isogonal with respect to $\measuredangle X_1XX_2$.
   – $XF_i$ bisects $\measuredangle X_1F_iX_2$ where $i = 1$ or 2.
   – The normals and the tangents at $X_i$ bisect $\measuredangle F_1X_iF_2$ where $i = 1$ or 2.
   – The reflection of $F_i$ over $t_i$ is collinear with the other focus and one tangency point.

   Also for every pair of isogonal conjugates, a conic tangent to all three sides of $\triangle ABC$ and having them as the foci exists.

   *Proof.* Exercise. Use a good characterisation of the tangent to the conic (sum and difference of distances). □

## • Some useful isogonal conjugates

| **Point** | **Isogonal Conjugate** |
|---|---|
| Orthocenter | Circumcenter |
| Centroid | Point of concurrence of symmedians |
| Gergonne Point | Insimilicenter of the circumcircle and the incircle |
| Nagel Point | Exsimilicenter of the circumcircle and the incircle |
| In/Excenters | Themselves |
| Nine point center | Kosnita point |

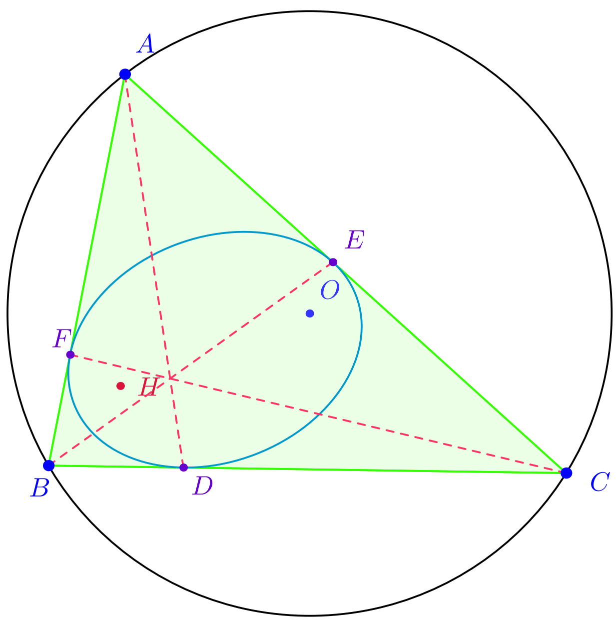
For more, see the Kimberling Encyclopedia of Triangle Centers.

# § Examples

1. Symmedians lemma

   *Proof.* Let the reflection of $A$ in $M$ be $D$. Then $BD, BX$ are isogonal and $CD, CX$ are isogonal. Thus, $D$ and $X$ are isogonal conjugates, and so $AD$ and $AX$ are isogonal, as desired. □

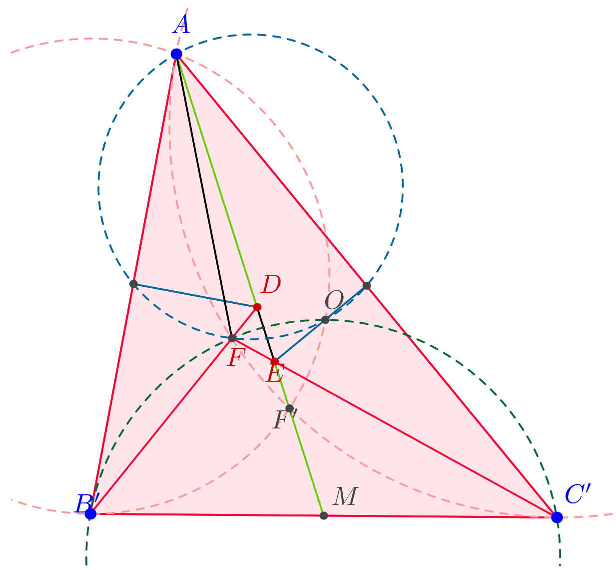
2. (IMO 2000 SL G3): Do there exist points $D$, $E$, $F$ on $BC, CA, AB$ of an acute triangle $\triangle ABC$ respectively such that $AD, BE, CF$ concur and $OD + DH = OE + EH = OF + FH = R$?




(2)

   *Proof.* Yes, there exist such points.
   Consider the conic with foci $O$, $H$ and tangent to the sides of $\triangle ABC$. Since the reflections of $H$ over the sides lie on the circumcircle, for any point $X$ of the ellipse, $OX + XH = R$. Now Brianchon's theorem finishes the problem. □

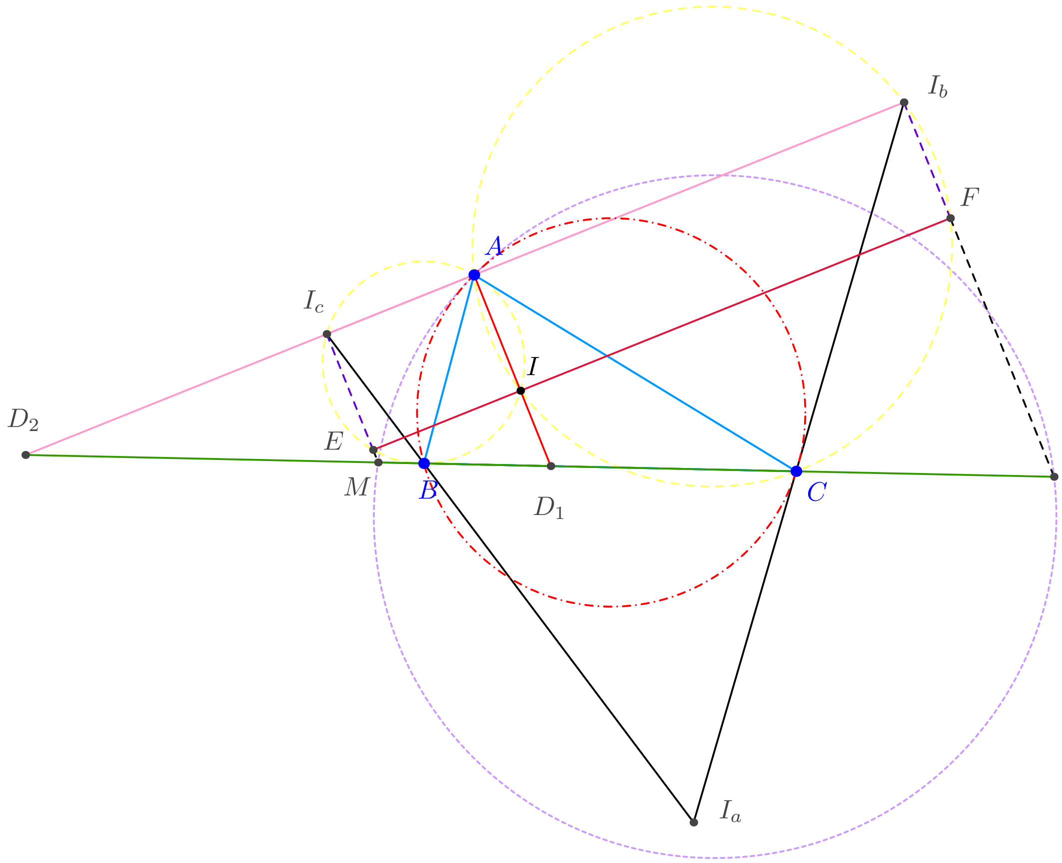
3. (USAMO 2008 P2): Let the $A-$median of a triangle $ABC$ be $AM$, and let the perpendicular bisectors of $AB$ and $AC$ meet $AM$ at $D$ and $E$. $BD$ and $CE$ intersect at $F$. Prove that $A, F, O$ and the midpoints of $AB$ and $AC$ are concyclic.




(3)

*Proof.* Note that $\measuredangle BFC = 2\measuredangle A = \measuredangle BOC$ and so $B, F, O, C$ lie on a circle. We need that the circle with $AO$ as diameter and $(BOC)$ meet at $F$, for which it suffices to show that $AF$ is a symmedian. Consider the isogonal conjugate of $F$, say $F^*$. By the angle conditions, both $(AF^*B)$ and $(AF^*C)$ are tangent to $BC$, and thus by radical axis, $AF^*$ is a median, which completes the proof. $\square$

The point $F^*$ is sometimes called the $A-$humpty point (Warning: this notation is not at all standard). We shall encounter this point later too.




(4)

4. (AoPS): Let $I$ be the incenter of $ABC$ and let $l$ be the line through $I$ and perpendicular to $AI$. The perpendicular to $AB$ through $B$ and that to $AC$ through $C$ meet $l$ at $E$ and $F$ respectively. The feet of perpendiculars from $E$ and $F$ to $l$ onto $BC$ are $M$ and $N$. Prove that $(AMN)$ and $(ABC)$ are tangent together.

   *Proof.* Firstly note that if $I_b, I_c$ are the excenters opposite $B$ and $C$, then $B, A, E$ lie on the circle with $II_b$ as diameter and so $I_b, E, M$ are collinear by angle chasing.In fact, $I_bI_cFE$ is a rectangle.
   If the external and the internal bisectors of $\measuredangle BAC$ meet $BC$ at $D_2, D_1$ respectively, then, $-1 = (B, C; D_1, D_2) \overset{I_a}{=} (I_c, I_b; A, D_2) \overset{P_{\infty \parallel AI}}{=} (N, M; D_1, D_2)$ and now since $AD_1 \perp AD_2$, they are the bisectors of $\measuredangle MAN$, from which it follows that $AM$ and $AN$ are isogonal cevians, from which the conclusion follows. □

# § Problems

The problems here are not sorted by difficulty; by easy we mean less interesting problems, by hard we mean interesting ones.

- ## Easy ones

1. Complete all the proofs left as exercises.

2. Prove that $IO = IH$ if and only if one of the angles of the triangle is $60°$.

3. Prove that the isogonal of $AP$ wrt $\measuredangle BPC$ and that of $AP^*$ wrt $\measuredangle BP^*C$ are symmetric wrt $BC$. (In fact, they meet on the tangency point of some special conic with $BC$).

4. Prove the following facts about the $A-$Humpty point:

   - Circles tangent to $BC$ and passing through $A, B$ and $A, C$ respectively, the circle with $AH$ as diameter, the circumcircle of $BHC$, the $A-$median, and the circumcircle of the triangle formed by the midpoint of $AH$, the reflection of $O$ in $BC$ and the midpoint of $BC$ all pass through the $A-$Humpty point.
   - It is the inverse of the midpoint of $BC$ under the inversion with radius $\sqrt{AH \cdot AD}$ and center $A$ where $H$ is the orthocenter and $AD$ is an altitude in $ABC$.

5. Find the complex and the barycentric coordinates of the isogonal conjugate of an arbitrary point $P$, assuming that the circumcircle is the unit circle.

- ## Medium Problems

1. Let $O$ be the circumcenter of $ABC$ and let the tangents to the circumcircle at $A, B, C$ form a triangle $XYZ$, and let the orthic triangle of $ABC$ be $DEF$. Prove that the isogonal conjugate of $O$ wrt $DEF$ is the orthocenter of $XYZ$.

2. Let the points of intersection of $AP, BP, CP$ with $BC, CA, AB$ be $A', B', C'$ respectively. Then prove that if the isogonal conjugate of $P$ wrt $ABC$ is $Q$ then the reflections of $AQ, BQ, CQ$ in $B'C', C'A', A'B'$ respectively concur at a point.

3. Let $DEF$ be the pedal triangle of $P$ wrt $ABC$, and let the points $X, Y, Z$ be on $PD, PE, PF$ respectively such that $PD \cdot PX = PE \cdot PY = PF \cdot PZ$. Then prove that $AX, BY, CZ$ concur at a point whose isogonal conjugate is on the line $OP^*$.

4. Let $K$ be the symmedian point of $ABC$, the isogonal conjugate of $G$. Prove the following properties:

   (a) It is the unique point which is the centroid of its pedal triangle.

(b) The midpoint of the $A-$altitude, the midpoint of $BC$ and $K$ are collinear.

(c) (Lemoine circle): Lines parallel to the sides of the triangle through $K$ are drawn. Prove that the points of intersections of these parallels with the sides are concyclic.

(d) (Another Lemoine circle): Lines antiparallel to the sides of the triangle through $K$ are drawn, meeting the remaining sides in $U, V, W, X, Y, Z$. Prove that these points are concyclic.

(e) The symmedians meet $(ABC)$ at $K_a, K_b, K_c$. Prove that $K$ is the symmedian point of the triangle formed by these points too.

- **Harder tasks**

1. (EMMO 2016[1]) Let $ABC$ be a triangle, the orthocenter of whose intouch triangle is $P$. The reflections of $P$ over the perpendicular bisectors of $BC, CA, AB$ are $X, Y, Z$, and the midpoints of $YZ, ZX, XY, BC, CA, AB$ are $D, E, F, A_1, B_1, C_1$. Prove that $A_1D, B_1E, C_1F$ concur at the radical center of the nine-point circles of $I_aBC, I_bCA, I_cAB$, where $I_a, I_b, I_c$ are the excenters of the triangle $ABC$, opposite to $A, B, C$ respectively.

2. (ELMO 2016, wording modified[2]) Let $ABC$ be a triangle and let $DEF$ be the intouch triangle, $AD, BE, CF$ being cevians. $AI$ meets $DE, DF$ at $M, N$ respectively. $S$ and $T$ are points on $BC$ such that $\measuredangle MSN = \measuredangle MTN = 90°$. Prove that:

   (a) $(AST)$ is tangent to the circumcircle of $ABC$.

   (b) $(AST)$ is tangent to the incircle of $ABC$.

   (c) [3] $(AST)$ is tangent to the $A-$excircle of $ABC$.

[1] This year's test was named "Every Mathematician Must Outperform"
[2] This was "Elmo Lives Mostly Outside"
[3] This wasn't part of the original problem, but looks good.

# § Hints to harder tasks

1. - For any general $P$, prove that the concurrency point is actually the complement of the isogonal conjugate of $P$, where the complement of any point is its image under the homothety $\mathbb{H}(G, \frac{-1}{2})$.
   - Then prove that the isogonal conjugate of the given point $P$ is the concurrency point of the Euler lines of $IBC, ICA, IAB$, known as the Schiffler point.
   - To finish, prove that the complement of the Schiffler point is the radical center of the given nine-point circles.
2. Use isogonal lines, inversion, etc, etc, etc.

# § References

[1] Grinberg D., *Isogonal conjugation with respect to a triangle*, 23.09.2006

[2] Various posts on `www.artofproblemsolving.com`.