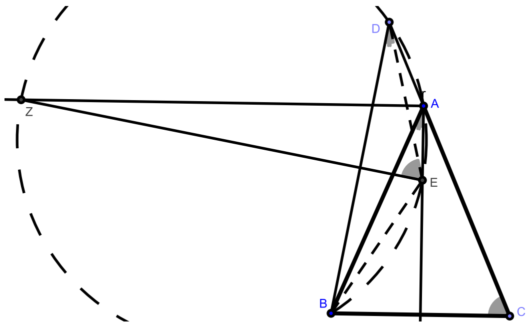
## 0.1 Geometry

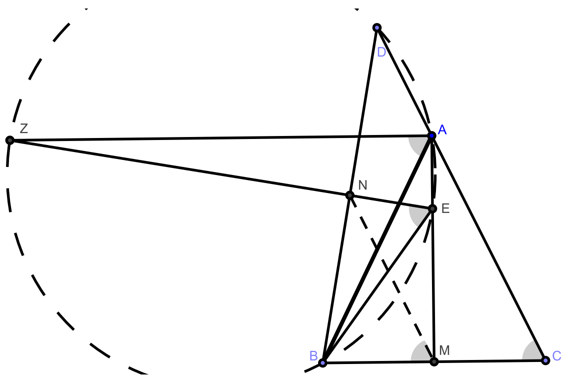
**G1** Let $ABC$ be an isosceles triangle with $AB = AC$. On the extension of the side $[CA]$ we consider the point $D$ such that $AD < AC$. The perpendicular bisector of the segment $[BD]$ meets the internal and the external bisectors of the angle $\widehat{BAC}$ at the points $E$ and $Z$, respectively. Prove that the points $A, E, D, Z$ are concyclic.

**Solution 1**




In $\triangle ABD$ the ray $[AZ$ bisects the angle $\widehat{DAB}$ and the line $ZE$ is the perpendicular bisector of the side $[BD]$. Hence $Z$ belongs to the circumcircle of $\triangle ABD$.
Therefore the points $A, B, D, Z$ are concyclic.
In $\triangle BCD$, $AE$ and $ZE$ are the perpendicular bisectors $[BC]$ and $[BD]$, respectively.
Hence, $E$ is the circumcenter of $\triangle BCD$ and therefore $\widehat{DEZ} = \widehat{BED}/2 = \widehat{ACB}$.
Since $BD \perp ZE$, we conclude that: $\widehat{BDE} = 90^\circ - \widehat{DEZ} = 90^\circ - \widehat{ACB} = \widehat{BAE}$.
Hence the quadrilateral $AEBD$ is cyclic, that is the points $A, B, D, E$ are concyclic.
Therefore, since $A, B, D, Z$ are also concyclic, we conclude that $AEZD$ is cyclic.

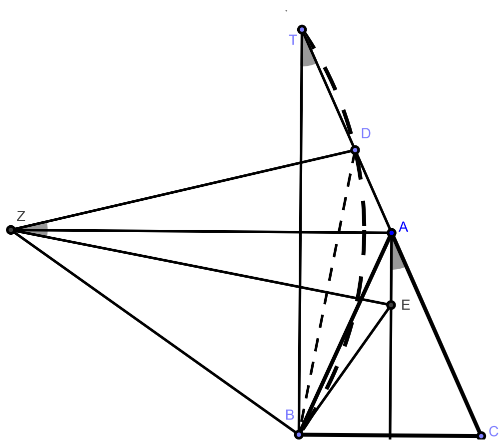
**Solution 2**




In $\triangle ABD$ the ray $[AZ$ bisects the angle $\widehat{DAB}$ and the line $ZE$ is the perpendicular bisector of the side $[BD]$. Hence $Z$ belongs to the circumcircle of $\triangle ABD$.
Therefore the points $A, B, D, Z$ are concyclic.

Let $M$ and $N$ be the midpoints of the sides $[BC]$ and $[DB]$, respectively. Then $N \in ZE$ and $M \in AE$. Next, $[MN]$ is a midline in $\triangle BCD$, so $MN \parallel CD \Rightarrow \widehat{NMB} \equiv \widehat{ACB}$.
But $[AZ$ is the external bisector of the angle $\widehat{BAC}$ of $\triangle ABC$, hence $\widehat{BAZ} \equiv \widehat{ACB}$.
Therefore, $\widehat{NMB} \equiv \widehat{BAZ}$. In the quadrilateral $BMEN$ we have $\widehat{BNE} = \widehat{BME} = 90°$, so $BMEN$ is cyclic $\Rightarrow \widehat{NMB} \equiv \widehat{BEZ}$, hence $\widehat{BAZ} \equiv \widehat{BEZ} \Rightarrow AEBZ$ is cyclic.
Therefore, since $A, B, D, Z$ are also concyclic, we conclude that $AEZD$ is cyclic.

**Solution 3**

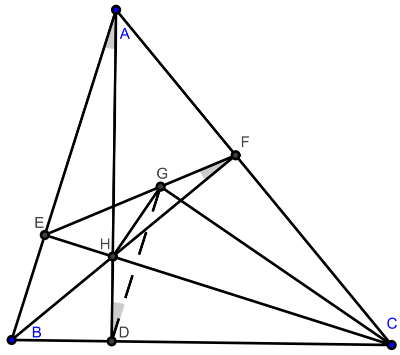



Let $T$ be the symmetric of $B$ with respect to the axis $AZ$. Obviously $T \in AD$. Since $AE$ and $BT$ are both perpendiculars to $AZ$, they are parallel, so $\widehat{BAC} \equiv \widehat{BTA}$. (1)
Since $ZB = ZT = ZD$, the point $Z$ is the circumcenter of $\triangle BDT$.
Therefore $\widehat{BTA} = \widehat{BZD}/2 = \widehat{EZD}$. (2)
From (1) and (2) we conclude that $\widehat{EAC} \equiv \widehat{EZD}$, which gives that $AEDZ$ is cyclic.

**G2** Let $AD$, $BF$ and $CE$ be the altitudes of $\triangle ABC$. A line passing through $D$ and parallel to $AB$ intersects the line $EF$ at the point $G$. If $H$ is the orthocenter of $\triangle ABC$, find the angle $\widehat{CGH}$.

**Solution 1**

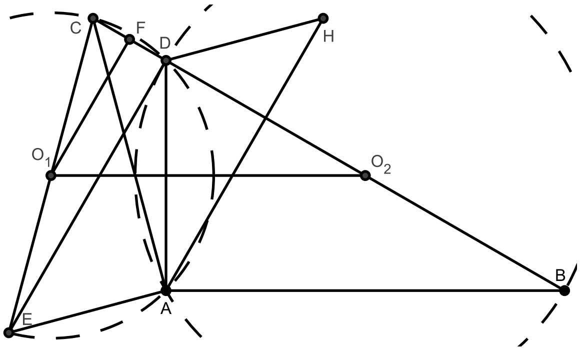
We can see easily that points $C, D, H, F$ lies on a circle of diameter $[CH]$.
Take $\{F, G'\} = \odot(CHF) \cap EF$. We have $\widehat{EFH} = \widehat{BAD} = \widehat{BCE} = \widehat{DFH}$ since the quadrilaterals $AEDC$, $AEHF$, $CDHF$ are cyclic. Hence $[FB$ is the bisector of $\widehat{EFD}$, so $H$ is the midpoint of the arc $DG'$. It follows that $DG' \perp CH$ since $[CH]$ is a diameter. Therefore $DG' \parallel AB$ and $G \equiv G'$. Finally $G$ lies on the circle $\odot(CFH)$, so $\widehat{HGC} = 90°$.




**Solution 2**
The quadrilateral $AEHF$ is cyclic since $\widehat{AEH} = \widehat{AFH} = 90°$, so $\widehat{EAD} \equiv \widehat{GFH}$.
But $AB \parallel GD$, hence $\widehat{EAD} \equiv \widehat{GDH}$. Therefore $\widehat{GFH} \equiv \widehat{GDH} \Rightarrow DFGH$ is cyclic.
Because the quadrilateral $CDHF$ is cyclic since $\widehat{CDH} = \widehat{CFH} = 90°$, we conclude that the quadrilateral $CFGH$ is cyclic, which gives that $\widehat{CGH} = \widehat{CFH} = 90°$.

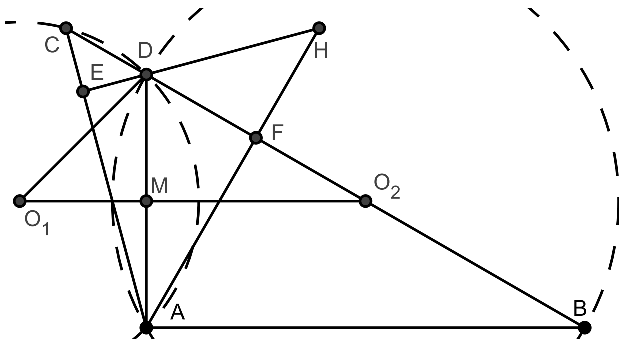
**G3** Let $ABC$ be a triangle in which $(BL$ is the angle bisector of $\widehat{ABC}$ $(L \in AC)$, $AH$ is an altitude of $\triangle ABC$ $(H \in BC)$ and $M$ is the midpoint of the side $[AB]$. It is known that the midpoints of the segments $[BL]$ and $[MH]$ coincides. Determine the internal angles of triangle $\triangle ABC$.
**Solution**
Let $N$ be the intersection of the segments $[BL]$ and $[MH]$. Because $N$ is the midpoint of both segments $[BL]$ and $[MH]$, it follows that $BMLH$ is a parallelogram. This implies that $ML \parallel BC$ and $LH \parallel AB$ and hence, since $M$ is the midpoint of $[AB]$, the angle bisector $[BL$ and the altitude $AH$ are also medians of $\triangle ABC$. This shows that $\triangle ABC$ is an equilateral one with all internal angles measuring $60°$.

**G4** Point $D$ lies on the side $[BC]$ of $\triangle ABC$. The circumcenters of $\triangle ADC$ and $\triangle BAD$ are $O_1$ and $O_2$, respectively and $O_1O_2 \parallel AB$. The orthocenter of $\triangle ADC$ is $H$ and $AH = O_1O_2$. Find the angles of $\triangle ABC$ if $2m(< C) = 3m(< B)$.
**Solution 1**



As $AD$ is the radical axis of the circumcircles of $\triangle ADC$ and $\triangle BAD$, we have that $O_1O_2 \perp AD$, therefore $\widehat{DAB} = 90°$. Let $F$ be the midpoint of $[CD]$ and $[CE]$ be a diameter of the circumcircle of $\triangle ADC$. Then $ED \perp CD$ and $EA \perp CA$, so $ED \parallel AH$ and $EA \parallel DH$ since $AH \perp CD$ and $DH \perp AC$ ($H$ is the orthocenter of $\triangle ADC$) and hence $EAHD$ is a parallelogram. Therefore $O_1O_2 = AH = ED = 2O_1F$, so in $\triangle FO_1O_2$ with $\widehat{O_1FO_2} = 90°$ we have $O_1O_2 = 2FO_1 \Rightarrow \widehat{ABC} = \widehat{O_1O_2F} = 30°$.
Then we get $\widehat{ACB} = 45°$ and $\widehat{BAC} = 105°$.

**Solution 2**

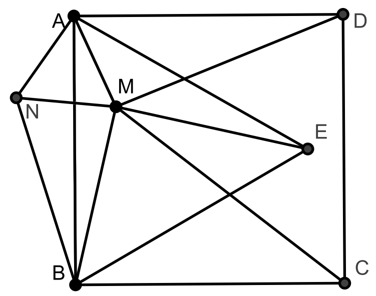



As $AD$ is the radical axis of the circumcircles of $\triangle ADC$ and $\triangle BAD$, we have that $O_1O_2 \perp AD$, therefore $\widehat{DAB} = 90°$ and $O_2$ is the midpoint of $[BD]$.
Take $\{E\} = DH \cap AC,\ \{F\} = AH \cap BC$ and $\{M\} = AD \cap O_1O_2$.
We have $\widehat{CEH} = \widehat{CFH} = 90° \Rightarrow CEFH$ is cyclic, hence $\widehat{ACD} = \widehat{AHD}$.
But $\widehat{ACD} = arc\ AD/2 = \widehat{DO_1O_2}$, so $\widehat{AHD} = \widehat{DO_1O_2}$. We know that $AH = O_1O_2$.
We also have $\widehat{DAH} = \widehat{O_1O_2D}$ since $AO_2FM$ is cyclic with $\widehat{AMO_2} = \widehat{AFO_2}$.
Therefore $\triangle HDA \equiv \triangle O_1DO_2 \Rightarrow DA = DO_2 = BD/2$, so in right-angled $\triangle ABD$ we have $m(\widehat{ABD}) = 30°$. Then we get $\widehat{ACB} = 45°$ and $\widehat{BAC} = 105°$.

**G5** Inside the square $ABCD$, the equilateral triangle $\triangle ABE$ is constructed. Let $M$ be an interior point of the triangle $\triangle ABE$ such that $MB = \sqrt{2},\ MC = \sqrt{6},\ MD = \sqrt{5}$ and $ME = \sqrt{3}$. Find the area of the square $ABCD$.

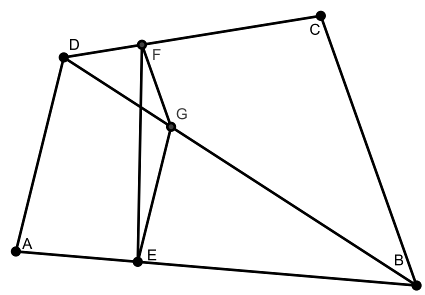
**Solution**

Let $K, F, H, Z$ be the projections of point $M$ on the sides of the square.
Then by **Pythagorean Theorem** we can prove that $MA^2 + MC^2 = MB^2 + MD^2$.
From the given condition we obtain $MA = 1$.
With center $A$ and angle $60°$, we rotate $\triangle AME$, so we construct the triangle $ANB$.




Since $AM = AN$ and $\widehat{MAN} = 60°$, it follows that $\triangle AMN$ is equilateral and $MN = 1$.
Hence $\triangle BMN$ is right-angled because $BM^2 + MN^2 = BN^2$.
So $m(\widehat{BMA}) = m(\widehat{BMN}) + m(\widehat{AMN}) = 150°$.
Applying **Pythagorean Generalized Theorem** in $\triangle AMB$, we get:
$AB^2 = AM^2 + BM^2 - 2AM \cdot BM \cdot \cos 150° = 1 + 2 + 2\sqrt{2} \cdot \sqrt{3} : 2 = 3 + \sqrt{6}$.
We conclude that the area of the square $ABCD$ is $3 + \sqrt{6}$.

**G6** Let $ABCD$ be a convex quadrilateral, $E$ and $F$ points on the sides $AB$ and $CD$, respectively, such that $\frac{AB}{AE} = \frac{CD}{DF} = n$. Denote by $S$ the area of the quadrilateral $AEFD$. Prove that $S \leq \frac{AB \cdot CD + n(n-1) \cdot DA^2 + n \cdot AD \cdot BC}{2n^2}$.
**Solution**




By **Ptolemy's Inequality** in $AEFD$, we get $S = \frac{AF \cdot DE \cdot \sin(\widehat{AF, DE})}{2} \leq \frac{AF \cdot DE}{2} \leq \frac{AE \cdot DF + AD \cdot EF}{2} = \frac{AB \cdot CD + n^2 \cdot DA \cdot EF}{2n^2}$.
Let $G$ be a point on diagonal $BD$ such that $\frac{DB}{DG} = n$. By **Thales's Theorem** we get $GE = \frac{(n-1)AD}{n}$ and $GF = \frac{BC}{n}$. Applying the inequality of triangle in $\triangle EGF$ we get $EF \leq EG + GF = \frac{(n-1)AD + BC}{n}$. Now, we get:
$S \leq \frac{AB \cdot CD + n^2 AD \cdot EF}{2n^2} \leq \frac{AB \cdot CD + n(n-1) \cdot DA^2 + n \cdot AD \cdot BC}{2n^2}$.