The Mathematical Danube Competition
Călăraşi, October 29, 2016

Problem 1. Let $S = x_1x_2 + x_3x_4 + \ldots + x_{2015}x_{2016}$, where $x_1, x_2, \ldots, x_{2016} \in \{\sqrt{3}-\sqrt{2},\ \sqrt{3}+\sqrt{2}\}$. Is the equality $S = 2016$ possible?

Cristian Lazăr

Solution: The answer is in the affirmative.
The terms of the sum can be: $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2}) = 1$, $(\sqrt{3}+\sqrt{2})^2 = 5+2\sqrt{6}$ or $(\sqrt{3}-\sqrt{2})^2 = 5-2\sqrt{6}$. If there are a terms equal to 1, b terms equal to $5+2\sqrt{6}$ and c terms equal to $5-2\sqrt{6}$, then a, b, c need to satisfy $a+b+c = 1008$, $a+(5+2\sqrt{6})b+(5-2\sqrt{6})c = 2016$. The last equality can be written $a+5b+5c-2016 = \sqrt{6}(2c-2b)$. As $\sqrt{6}$ is irrational, it follows that $b = c$ and $a+5b+5c = 2016$. Finally we obtain $a = 756$, $b = c = 126$.

Problem 2. Determine the positive integers $n > 1$ such that, for any divisor d of n, the numbers d^2-d+1 and d^2+d+1 are prime.

Lucian Petrescu

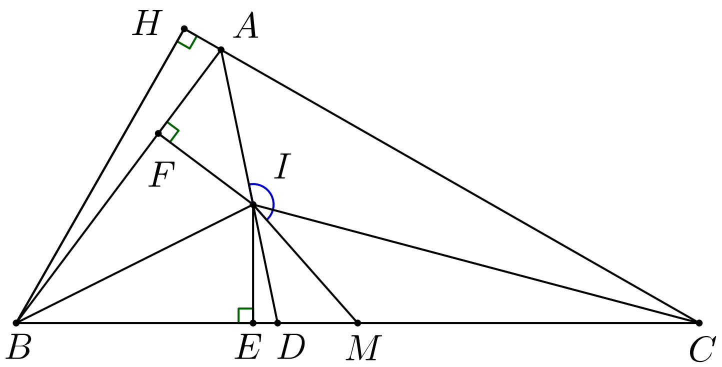
Solution: The answer is: $n \in \{2,3,6\}$.
First, we prove that n is square-free. If d^2 divides n for a positive integer $d > 1$, then $(d^2)^2+d^2+1$ would be a prime number. But $d^4+d^2+1 = (d^2-d+1)(d^2+d+1)$, with both factors larger than 1, which is a contradiction.
Thus, $n = p_1 \cdot p_2 \cdot \ldots \cdot p_s$, where $s \in \mathbb{N}$ and $p_1 < p_2 < \ldots < p_s$ are prime numbers.
Let $p > 5$ be a prime number. Then $p \equiv 1 \pmod 6$ or $p \equiv 5 \pmod 6$. If $p \equiv 1 \pmod 6$, then $p^2+p+1 \equiv 3 \pmod 6$, and $p^2+p+1 > 3$ is composite.
If $p \equiv 5 \pmod 6$, then $p^2-p+1 \equiv 3 \pmod 6$, and $p^2-p+1 > 3$ is composite.
In conclusion, the only prime factors of n can be 2 and 3, so $n \in \{2,3,6\}$. It is easy to check that all these three numbers fulfill the given condition.

Problem 3. Let ABC be a triangle with $AB < AC$, I its incenter, and M the midpoint of the side BC. If $IA = IM$, determine the smallest possible value of the angle AIM.
Solution: Answer: $150°$.
Let $\{D\} = AI \cap BC$. As $AB < AC$, D lies between B and M and $\angle ACB < \angle ABC$.
We have $\angle IDB = \angle DAC + \angle ACB < \angle DAB + \angle ABD = \angle ADC$, therefore angle IDB is acute.
Let F and E be the projections of I onto AB and BC, respectively. It follows that $E \in (BD) \subset BM$. Triangles IBF and IBE are congruent and so are triangles IFA and IEM, therefore $BA = BM = \dfrac{BC}{2}$ and triangles IBA and IBM are congruent.
We have: $\angle MID = \angle IDB - \angle IMB = \angle DAC + \angle ACD - \angle IAB = \angle ACD$. It

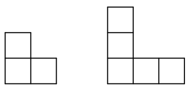
follows that $\angle AIM = 180° - \angle ACB$ (1).

Let H be the projection of B onto the line AC. It follows that $BH \leq AB = \dfrac{BC}{2}$, which shows that $\angle ACB \leq 30°$ (2).

From (1) and (2) we obtain that $\angle AIM \geq 180° - 30° = 150°$.

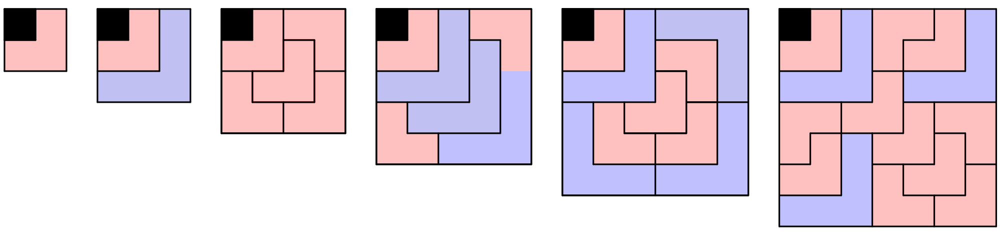


Problem 4. A unit square is removed from the corner of the $n \times n$ grid where $n \geq 2$. Prove that the remainder can be covered by copies of the „L-shapes" consisting of 3 or 5 unit squares depicted in the figure. Every square must be covered once and the L-shapes must not go over the bounds of the grid.



Estonian Olympiad, 2009

Solution: Without loss of generality, we may assume that the unit square that has been removed is the one in the top left corner. We call such a grid an n-grid. We prove the assertion by an induction of step 6. The examples below show that an n-grid with $n \in \{2, 3, 4, 5, 6, 7\}$ can be covered with L-shapes.



It is easy to see that we can glue together two L-shapes consisting of 3 unit squares in order to obtain a 2×3 or a 3×2 rectangle. From several such rectangles we can form 2×6 and 3×6 rectangles. Combining these, we can obtain any $m \times 6$ rectangle with $m > 1$. Now to the induction step. If an n-grid can be covered, so can an $(n+6)$-grid. Indeed, decompose the $(n+6)$-grid into an n-grid positioned in its upper left corner, an $(n+6) \times 6$ rectangle consisting of the last 6 columns, and the remaining part, a $6 \times n$ rectangle. Each of these pieces can be covered, therefore so can the $(n+6)$-grid. Our induction is thus complete.