Problemele de combinatorică din lista scurtă de la Balcaniada de juniori, 2010-2013

2010

C1 There are two piles of coins, each containing 2010 pieces. Two players A and B play a game taking turns (A plays first). At each turn, the player on play has to take one or more coins from one pile or exactly one coin from each pile. Whoever takes the last coin is the winner. Which player will win if they both play in the best possible way?

C2 A 9×7 rectangle is tiled with tiles of the two types: L-shaped tiles composed by three unit squares (can be rotated repeatedly with $90°$) and square tiles composed by four unit squares. Let $n \geq 0$ be the number of the 2×2 tiles which can be used in such a tiling. Find all the values of n.

2011

C1 Inside of a square whose side length is 1 there are a few circles such that the sum of their circumferences is equal to 10. Show that there exists a line that meets at least four of these circles.

C2 Can we divide an equilateral triangle $\triangle ABC$ into 2011 small triangles using 122 straight lines? (there should be 2011 triangles that are not themselves divided into smaller parts and there should be no polygons which are not triangles)

C3 We can change a natural number n in three ways:
a) If the number n has at least two digits, we erase the last digit and we subtract that digit from the remaining number (for example, from 123 we get $12 - 3 = 9$);
b) If the last digit is different from 0, we can change the order of the digits in the opposite one (for example, from 123 we get 321);
c) We can multiply the number n by a number from the set $\{1, 2, 3, ..., 2010\}$.
Can we get the number 21062011 from the number 1012011?

C4 In a group of n people, each one had a different ball. They performed a sequence of swaps; in each swap, two people swapped the ball they had at that moment. Each pair of people performed at least one swap. In the end each person had the ball he/she had at the start. Find the least possible number of swaps, if: *a)* $n = 5$; *b)* $n = 6$. (Ivaylo Kortezov, Bulgaria)

C5 A set S of natural numbers is called *good*, if for each element $x \in S$, x does not divide the sum of the remaining numbers in S. Find the maximal possible number of elements of a *good* set which is a subset of the set $A = \{1, 2, 3, ..., 63\}$.

C6 Let $n > 3$ be a positive integer. An equilateral triangle $\triangle ABC$ is divided into n^2 smaller congruent equilateral triangles (with sides parallel to its sides).
Let m be the number of rhombuses that contain two small equilateral triangles and d the number of rhombuses that contain eight small equilateral triangles. Find the difference $m - d$ in terms of n.

C7 Consider a rectangle whose lengths of sides are natural numbers. If someone places as many squares as possible, each with area 3, inside of the given rectangle, such that the sides of the squares are parallel to the rectangle sides, then the maximal number of these squares fill exactly half of the area of the rectangle. Determine the dimensions of all rectangles with this property.

C8 Determine the polygons with n sides, $n \geq 4$, not necessarily convex, that satisfy the property that the reflection of every vertex of the polygon with respect to every diagonal of the polygon does not fall outside the polygon.
Note. A diagonal is any segment joining two non-neighbouring vertices of the polygon; the reflection is considered with respect to the support line of the diagonal.
(România)

C9 Decide if it is possible to consider 2011 points in a plane such that:
i) the distance between every two of these points is different from 1, and
ii) every unit circle centered at one of these points leaves exactly 1005 of the points outside the circle. (România)

2012

C1 Along a round table are arranged 11 cards with the names (all distinct) of the 11 members of the 16^{th} JBMO Problem Selection Commttee. The distances between each two consecutive cards are equal. Assume that in the first meeting of the Committee none of its 11 members sits in frint of the card with his name. Is it possible to rotate the table by some angle so that at the end at least two members sit in front of the card with their names?

C2[1] On a board there are n nails, each two connected by a rope. Each rope is colored in one of n given distinct colors. For each three distinct colors, there exist three nails connected with ropes of these three colors.
a) Can n be 6 ?
b) Can n be 7 ?

C3 In a circle of diameter 1 consider 65 points no three of which are colinear. Prove that there exist 3 among these points which form a triangle with area less

[1] pentru $n = 10$ problema s-a dat în 1995 la olimpiadă în Rusia

then or equal to $\dfrac{1}{72}$.

2013

C1 Find the largest number of distinct integers that can be chosen from the set $\{1, 2, \ldots, 2013\}$ so that the difference of no two of them is equal to 17. (Macedonia)

C2 On a billiards table in the shape of a rectangle $ABCD$ with $AB = 2013$ and $AD = 1000$ a billiard ball is shot along the bisector of the angle $\sphericalangle BAD$. Assuming that the ball is reflected from the sides at the same angle it comes in, determine whether it will ever go to the corner B. (Bosnia şi Herţegovina)

C3 Let n be a positive integer. Two players, Alice and Bob, are playing the following game:
- Alice chooses n real numbers; not necessarily distinct.
- Alice writes all pairwise sums on a sheet of paper and gives it to Bob. (There are $\frac{n(n-1)}{2}$ such sums; not necessarily distinct.)
- Bob wins if he finds correctly the initial n numbers chosen by Alice with only one guess.
Can Bob be sure to win for the following cases?

a. $n = 5$
b. $n = 6$
c. $n = 8$

Justify your answer(s).

[For example, when $n = 4$, Alice may choose the numbers 1, 5, 7, 9, which have the same pairwise sums as the numbers 2, 4, 6, 10, and hence Bob cannot be sure to win.]

Enunţul iniţial:

Al possible pairs of n apples are weighted and the results are given to us in an arbitrary order. Can we determine the weights of the apples if **a.** $n = 4$, **b.** $n = 5$, **c.** $n = 6$? (Bulgaria)

Probleme conexe:

1. Câte submulţimi ale mulţimii $\{1, 2, \ldots, 2013\}$ au proprietatea că nu conţin elemente a căror diferenţă să fie 17 şi au r elemente, unde r este răspunsul de la problema C1/2013?

2. Câte submulţimi ale mulţimii $\{1, 2, \ldots, 2013\}$ au proprietatea că nu conţin elemente a căror diferenţă să fie 17? (legată de C1/2013)

3. Pe o masă sunt 2013 chibrituri. Doi oameni joacă după următoarele reguli: la fiecare mişcare se poate lua de pe masă un număr de chibrituri egal cu o putere oarecare a lui 2 (1,2,4,8...). Pierde acel jucător care nu mai poate efectua mişcări. Cine câştigă într-un joc corect, primul sau al doilea jucător?

(Marcel Teleucă, Şcoala cu ceas, 2013, legată de C1/2010)

4. A coin is placed in a lower-left corner of a chessboard. Players A and B play the following game: A starts the game, and on their turn a player moves the coin to a position that is immediately to the top, right, top-right, top-left, or down-right of the position where the coin is currently placed. The player who makes the last move wins. Who has the winning strategy, and what is the strategy? (legată de C1/2010)

5. A rectangular floor is covered by 2×2 and 1×4 tiles. One tile got smashed. There is a tile of the other kind available. Show that the floor cannot be covered by rearranging the tiles. (Engel, legată de C2/2010)

6. The term a_{n+1}, $n \in \mathbb{N}$, of a sequence of positive integers is defined as follows: the last digit of a_n is removed, multiplied by 4, then added to what is left of a_n. For example, if $a_n = 693$, then $a_{n+1} = 69 + 4 \cdot 3 = 81$, $a_{n+2} = 8 + 4 \cdot 1 = 12$, $a_{n+3} = 1 + 4 \cdot 2 = 9$, $a_{n+4} = 0 + 4 \cdot 9 = 36$, etc. Prove that if 1001 is a term of the sequence, then the sequence does not contain any prime numbers. (Olimpiadă Germania, legată de problema C3/2011)

7. Generalizaţi problema C2/2012 pentru un $n \in \mathbb{N}$, $n \geq 3$ arbitrar.

8. Consider n children in a playground, where $n \geq 2$. Every child has a coloured hat, and every pair of children is joined by a coloured ribbon. For every child, the colour of each ribbon held is diferent, and also different from the colour of that child's hat. What is the minimum number of colours that needs to be used? (PAMO 2009, legată de problema C2/2012)

9. Consider all the possible subsets of the set $\{1, 2, \ldots, N\}$ which do not contain any consecutive numbers. Prove that the sum of the squares of the products of the numbers in these subsets is $(N + 1)! - 1$. (Turneul Oraşelor, 1989, legată de problema C1/2013)