0.1 Combinatorics

C1 Inside of a square whose side length is 1 there are a few circles such that the sum of their circumferences is equal to 10. Show that there exists a line that meets alt least four of these circles.

Solution

Find projections of all given circles on one of the sides of the square. The projection of each circle is a segment whose length is equal to the length of a diameter of this circle. Since the sum of the lengths of all circles' diameters is equal to $10/\pi$, it follows that the sum of the lengths of all mentioned projections is equal to $10/\pi > 3$. Because the side of the square is equal to 1, we conclude that at least one point is covered with at least four of these projections. Hence, a perpendicular line to the projection side passing through this point meets at least four of the given circles, so this is a line with the desired property.

C2 Can we divide an equilateral triangle $\triangle ABC$ into 2011 small triangles using 122 straight lines? (there should be 2011 triangles that are not themselves divided into smaller parts and there should be no polygons which are not triangles)

Solution

Firstly, for each side of the triangle, we draw 37 equidistant, parallel lines to it. In this way we get $38^2 = 1444$ triangles. Then we erase 11 lines which are closest to the vertex A and parallel to the side BC and we draw 21 lines perpendicular to BC, the first starting from the vertex A and 10 on each of the two sides, the lines which are closest to the vertex A, distributed symmetrically. In this way we get $26 \cdot 21 + 10 = 556$ new triangles. Therefore we obtain a total of 2000 triangles and we have used $37 \cdot 3 - 11 + 21 = 121$ lines. Let D be the 12^{th} point on side AB, starting from B (including it). The perpendicular to BC passing through D will be the last line we draw. In this way we obtain the required configuration.

C3 We can change a natural number n in three ways:

a) If the number n has at least two digits, we erase the last digit and we subtract that digit from the remaining number (for example, from 123 we get $12 - 3 = 9$);

b) If the last digit is different from 0, we can change the order of the digits in the opposite one (for example, from 123 we get 321);

c) We can multiply the number n by a number from the set $\{1, 2, 3, ..., 2010\}$.

Can we get the number 21062011 from the number 1012011?

Solution

The answer is *NO*. We will prove that if the first number is divisible by 11, then all the numbers which we can get from n, are divisible by 11. When we use *a)*, from the number $10a+b$, we will get the number $m=a-b=11a-n$, so $11 \mid m$ since $11 \mid n$. It's well-known that a number is divisible by 11 if and only if the difference between sum of digits on even places and sum of digits on odd places is divisible by 11. Hence, when we use b), from a number which is divisible by 11, we will get a number which is also divisible by 11. When we use c), the obtained number remains divisible by 11. So, the answer is *NO* since 1012011 is divisible by 11 and 21062011 is not.

C4 In a group of n people, each one had a different ball. They performed a sequence of swaps; in each swap, two people swapped the ball they had at that moment. Each pair of people performed at least one swap. In the end each person had the ball he/she had at the start. Find the least possible number of swaps, if: *a)* $n=5$; *b)* $n=6$.

Solution

We will denote the people by $A, B, C, ...$ and their initial balls by the corresponding small letters.Thus the initial state is $Aa, Bb, Cc, Dd, Ee(, Ff)$. A swap is denoted by the (capital) letters of the people involved.

a) Five people form 10 pairs, so at least 10 swaps are necessary.

In fact, 10 swaps are sufficient:

Swap AB, then BC, then CA; the state is now Aa, Bc, Cb, Dd, Ee.

Swap AD, then DE, then EA; the state is now Aa, Bc, Cb, De, Ed.

Swap BE, then CD; the state is now Aa, Bd, Ce, Db, Ec.

Swap BD, then CE; the state is now Aa, Bb, Cc, Dd, Ee.

All requirements are fulfilled now, so the answer is 10.

b) Six people form 15 pairs, so at least 15 swaps are necessary. We will prove that the final number of swaps must be even. Call a pair formed by a ball and a person *inverted* if letter of the ball lies after letter of the person in the alphabet. Let T be the number of *inverted* pairs; at the start we have $T=0$. Each swap changes T by 1, so it changes the parity of T. Since in the end $T=0$, the total number of swaps must be even. Hence, at least 16 swaps are necessary. In fact 16 swaps are sufficient:

Swap AB, then BC, then CA; the state is now Aa, Bc, Cb, Dd, Ee, Ff.

Swap AD, then DE, then EA; the state is now Aa, Bc, Cb, De, Ed, Ff.

Swap FB, then BE, then EF; the state is now Aa, Bd, Cb, De, Ec, Ff.

Swap FC, then CD, then DF; the state is now Aa, Bd, Ce, Db, Ec, Ff.

Swap BD, then CE, then twice AF, the state is now Aa, Bb, Cc, Dd, Ee, Ff.

All requirements are fulfilled now, so the answer is 16.

C5 A set S of natural numbers is called *good*, if for each element $x \in S$, x does not divide the sum of the remaining numbers in S. Find the maximal possible number of elements of a *good* set which is a subset of the set $A = \{1, 2, 3, ..., 63\}$.

Solution

Let set B be the *good* subset of A which have the maximum number of elements. We can easily see that the number 1 does not belong to B since 1 divides all natural numbers. Based on the property of divisibility, we know that x divides the sum of the remaining numbers if and only if x divides the sum of all numbers in the set B. If B has exactly 62 elements, than $B = \{2, 3, 4, ..., 62\}$, but this set can't be good since the sum of its elements is 2015 which is divisible by 5. Therefore B has at most 61 elements. Now we are looking for the set, whose elements does not divide their sum, so the best way to do that is making a sum of elements be a prime number. $2 + 3 + 4 + ... + 63 = 2015$ and if we remove the number 4, we will obtain the prime number 2011. Hence the set $B = \{2, 3, 5, 6, 7, ..., 63\}$ is a *good* one. We conclude that our number is 61.

C6 Let $n > 3$ be a positive integer. An equilateral triangle $\triangle ABC$ is divided into n^2 smaller congruent equilateral triangles (with sides parallel to its sides).

Let m be the number of rhombuses that contain two small equilateral triangles and d the number of rhombuses that contain eight small equilateral triangles. Find the difference $m - d$ in terms of n.

Solution

We will count the rhombuses having their large diagonal perpendicular to one side of the large triangle, then we will multiply by 3. So consider the horizontal side, and then the other $n-1$ horizontal dividing lines, plus a horizontal line through the apex. Index them by $1, 2, ..., n$, from top down (the basis remains unindexed). Then each indexed k line contains exactly k nodes of this triangular lattice.

The top vertex of a rhombus containing two small triangles, and with its large diagonal vertical, may (and has to) a node of this triangular lattice found on the top $n - 1$ lines, so there are $m = 3[1 + 2 + ... + (n - 1)] = \dfrac{3n(n-1)}{2}$ such rhombuses.

The top vertex of a rhombus containing eight small triangles, and with its large diagonal vertical, may (and has to) be a node of this triangular lattice found on the top $n - 3$ lines, so there are $d = 3[1 + 2 + \cdots + (n - 3)] = \dfrac{3(n-3)(n-2)}{2}$ such rhombuses.

Finally we have $m - d = \dfrac{3}{2} \cdot (n^2 - n - n^2 + 5n - 6) = 6n - 9$.

C7 Consider a rectangle whose lengths of sides are natural numbers. If someone places as many squares as possible, each with area 3, inside of the given rectangle, such that the sides of the squares are parallel to the rectangle sides, then the maximal number of these squares fill exactly half of the area of the rectangle. Determine the dimensions of all rectangles with this property.

Solution

Let $ABCD$ be a rectangle with $AB = m$ and $AD = n$ where m, n are natural numbers such that $m \geq n \geq 2$. Suppose that inside of the rectangle $ABCD$ is placed a rectangular lattice consisting of some identical squares whose areas are equals to 3, where k of them are placed along the side AB and l of them along the side AD.

The sum of areas of all of this squares is equal to $3kl$. Besides of the obvious conditions $k\sqrt{3} < m$ and $l\sqrt{3} \leq n$ ***(1)***, by the assumption of the maximality of the lattice consisting of these squares, we must have $(k+1)\sqrt{3} > m$ and $(l+1)\sqrt{3} > n$ ***(2)***.

The proposed problem is to determine all pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ with $m \geq n \geq 2$, for which the ratio $R_{m,n} = \dfrac{3kl}{mn}$ is equal to $0,5$ where k, l are natural numbers determined by the conditions ***(1)*** and ***(2)***.

Observe that for $n \geq 6$, using ***(2)***, we get $R_{m,n} = \dfrac{k\sqrt{3} \cdot l\sqrt{3}}{mn} > \dfrac{(m-\sqrt{3})(n-\sqrt{3})}{mn} = \left(1 - \dfrac{\sqrt{3}}{m}\right)\left(1 - \dfrac{\sqrt{3}}{n}\right) \geq \left(1 - \dfrac{\sqrt{3}}{6}\right)^2 = \dfrac{1}{2} + \dfrac{7}{12} - \dfrac{\sqrt{3}}{3} > \dfrac{1}{2} + \dfrac{\sqrt{48}}{12} - \dfrac{\sqrt{3}}{3} = 0,5.$

So, the condition $R_{m,n} = 0,5$ yields $n \leq 5$ or $n \in \{2, 3, 4, 5\}$. We have 4 possible cases:

Case 1: $n = 2$. Then $l = 1$ and thus as above we get $R_{m,2} = \dfrac{3k}{2m} > \dfrac{\sqrt{3} \cdot (m - \sqrt{3})}{2m} = \dfrac{\sqrt{3}}{2} \cdot \left(1 - \dfrac{\sqrt{3}}{m}\right)$, which is greater than $0,5$ for each $m > \dfrac{\sqrt{27}+3}{2} > \dfrac{5+3}{2} = 4$, hence $m \in \{2, 3, 4\}$. Direct calculations give $R_{2,2} = R_{2,4} = 0,75$ and $R_{2,3} = 0,5$.

Case 2: $n = 3$. Then $l = 1$ and thus as above we get $R_{m,3} = \dfrac{3k}{3m} > \dfrac{\sqrt{3} \cdot (m - \sqrt{3})}{3m} = \dfrac{\sqrt{3}}{3} \cdot \left(1 - \dfrac{\sqrt{3}}{m}\right)$, which is greater than $0,5$ for each $m > 4\sqrt{3} + 6 > 12$, hence $m \in \{3, 4, ..., 12\}$. Direct calculations give $R_{3,3} = 0,(3),\ R_{3,5} = 0,4,\ R_{3,7} = 4/7,\ R_{3,9} = 5/9,\ R_{3,11} = 6/11$ and $R_{3,4} = R_{3,6} = R_{3,8} = R_{3,10} = R_{3,12} = 0,5$.

Case 3: $n = 4$. Then $l = 2$ and thus as above we get $R_{m,4} = \dfrac{6k}{4m} > \dfrac{\sqrt{3} \cdot (m - \sqrt{3})}{2m} = \dfrac{\sqrt{3}}{2} \cdot \left(1 - \dfrac{\sqrt{3}}{m}\right)$, which is greater than $0,5$ for each $m > \dfrac{\sqrt{27}+3}{2} > \dfrac{5+3}{2} = 4$.

Hence $m = 4$ and a calculation gives $R_{4,4} = 0,75$.

Case 4: $n = 5$. Then $l = 2$ and thus as above we get $R_{m,5} = \dfrac{6k}{5m} > \dfrac{2\sqrt{3} \cdot (m - \sqrt{3})}{5m} =$

$\frac{2\sqrt{3}}{5} \cdot \left(1 - \frac{\sqrt{3}}{m}\right)$, which is greater than $0,5$ for each $m > \frac{12(4\sqrt{3}+5}{23} > \frac{12 \cdot 11}{23} > 6$, hence $m \in \{5,6\}$. Direct calculations give $R_{5,5} = 0,48$ and $R_{5,6} = 0,6$.
We conclude that: $R_{i,j} = 0,5$ for $(i,j) \in \{(2,3);(3,4);(,3,6);(3,8);(3,10);(3,12)\}$.
These pairs are the dimensions of all rectangles with desired property.

C8 Determine the polygons with n sides ($n \geq 4$), not necessarily convex, which satisfy the property that the reflection of every vertex of polygon with respect to every diagonal of the polygon does not fall outside the polygon.
Note: Each segment joining two non-neighboring vertices of the polygon is a diagonal. The reflection is considered with respect to the support line of the diagonal.
Solution
A polygon with this property has to be convex, otherwise we consider an edge of the convex hull of this set of vertices which is not an edge of this polygon. All the others vertices are situated in one of the half-planes determined by the support-line of this edge, therefore the reflections of the others vertices falls outside the polygon.
Now we choose a diagonal. It divides the polygon into two parts, $P1$ and $P2$. The reflection of $P1$ falls into the interior of $P2$ and viceversa. As a consequence, the diagonal is a symmetry axis for the polygon. Then every diagonal of the polygon bisects the angles of the polygon and this means that there are 4 vertices and the polygon is a rhombus. Each rhombus satisfies the desired condition.

C9 Decide if it is possible to consider 2011 points in a plane such that the distance between every two of these points is different from 1 and each unit circle centered at one of these points leaves exactly 1005 points outside the circle.
Solution
NO. If such a configuration existed, the number of segments starting from each of the 2011 points towards the other one and having length less than 1 would be 1005.
Since each segment is counted twice, their total number would be $1005 \cdot 2011/2$ which is not an integer, contradiction!