0.1 Algebra

A1 Let a, b, c be positive real numbers such that $abc = 1$. Prove that:

$$(a^5+a^4+a^3+a^2+a+1)(b^5+b^4+b^3+b^2+b+1)(c^5+c^4+c^3+c^2+c+1) \geq 8(a^2+a+1)(b^2+b+1)(c^2+c+1).$$

Solution
We have $x^5 + x^4 + x^3 + x^2 + x + 1 = (x^3 + 1)(x^2 + x + 1)$ for all $x \in \mathbb{R}_+$.
Take $S = (a^2 + a + 1)(b^2 + b + 1)(c^2 + c + 1)$.
The inequality becomes $S(a^3 + 1)(b^3 + 1)(c^3 + 1) \geq 8S$.
It remains to prove that $(a^3 + 1)(b^3 + 1)(c^3 + 1) \geq 8$.
By $AM - GM$ we have $x^3 + 1 \geq 2\sqrt{x^3}$ for all $x \in \mathbb{R}_+$.
So $(a^3 + 1)(b^3 + 1)(c^3 + 1) \geq 2^3 \cdot \sqrt{a^3b^3c^3} = 8$ and we are done.
Equality holds when $a = b = c = 1$.

A2 Let x, y, z be positive real numbers. Prove that:

$$\frac{x+2y}{z+2x+3y} + \frac{y+2z}{x+2y+3z} + \frac{z+2x}{y+2z+3x} \leq \frac{3}{2}.$$

Solution 1
Notice that $\sum_{cyc} \frac{x+2y}{z+2x+3y} = \sum_{cyc} \left(1 - \frac{x+y+z}{z+2x+3y}\right) = 3 - (x+y+z) \sum_{cyc} \frac{1}{z+2x+3y}$.
We have to proof that $3-(x+y+z) \sum_{cyc} \frac{1}{z+2x+3y} \leq \frac{3}{2}$ or $\frac{3}{2(x+y+z)} \leq \sum_{cyc} \frac{1}{z+2x+3y}$.
By *Cauchy-Schwarz* we obtain $\sum_{cyc} \frac{1}{z+2x+3y} \geq \frac{(1+1+1)^2}{\sum_{cyc}(z+2x+3y)} = \frac{3}{2(x+y+z)}$.

Solution 2
Because the inequality is homogenous, we can take $x + y + z = 1$.
Denote $x + 2y = a,\ y + 2z = b,\ z + 2x = c$. Hence, $a + b + c = 3(x + y + z) = 3$.
We have $(k-1)^2 \geq 0 \Leftrightarrow (k+1)^2 \geq 4k \Leftrightarrow \frac{k+1}{4} \geq \frac{k}{k+1}$ for all $k > 0$.
Hence $\sum_{cyc} \frac{x+2y}{z+2x+3y} = \sum \frac{a}{1+a} \leq \sum \frac{a+1}{4} = \frac{a+b+c+3}{4} = \frac{3}{2}$.

A3 Let a, b be positive real numbers. Prove that $\sqrt{\frac{a^2+ab+b^2}{3}} + \sqrt{ab} \leq a + b$.
Solution 1
Applying $x + y \leq \sqrt{2(x^2 + y^2)}$ for $x = \sqrt{\frac{a^2+ab+b^2}{3}}$ and $y = \sqrt{ab}$, we will obtain

$$\sqrt{\frac{a^2+ab+b^2}{3}} + \sqrt{ab} \leq \sqrt{\frac{2a^2+2ab+2b^2+6ab}{3}} \leq \sqrt{\frac{3(a^2+b^2+2ab)}{3}} = a + b.$$

Solution 2

The inequality is equivalent to
$\frac{a^2+ab+b^2}{3}+\frac{3ab}{3}+2\sqrt{\frac{ab(a^2+ab+b^2)}{3}}\le\frac{3a^2+6ab+3b^2}{3}$. This can be rewritten as $2\sqrt{\frac{ab(a^2+ab+b^2)}{3}}\le\frac{2(a^2+ab+b^2)}{3}$ or $\sqrt{ab}\le\sqrt{\frac{a^2+ab+b^2}{3}}$ which is obviously true since $a^2+b^2+ab\ge 2ab+ab=3ab$.

A4 Let x, y be positive real numbers such that $x^3+y^3\le x^2+y^2$. Find the greatest possible value of the product xy.

Solution 1

We have $(x+y)(x^2+y^2)\ge(x+y)(x^3+y^3)\ge(x^2+y^2)^2$, hence $x+y\ge x^2+y^2$. Now $2(x+y)\ge(1+1)(x^2+y^2)\ge(x+y)^2$, thus $2\ge x+y$. Because $x+y\ge 2\sqrt{xy}$, we will obtain $1\ge xy$. Equality holds when $x=y=1$.
So the greatest possible value of the product xy is 1.

Solution 2

By $AM-GM$ we have $x^3+y^3\ge\sqrt{xy}\cdot(x^2+y^2)$, hence $1\ge\sqrt{xy}$ since $x^2+y^2\ge x^3+y^3$. Equality holds when $x=y=1$. So the greatest possible value of the product xy is 1.

A5 Determine the positive integers a, b such that $a^2b^2+208=4\{lcm[a;b]+gcd(a;b)\}^2$.

Solution

Let $d=gcd(a,b)$ and $x,y\in\mathbb{Z}_+$ such that $a=dx$, $b=dy$. Obviously, $(x,y)=1$. The equation is equivalent to $d^4x^2y^2+208=4d^2(xy+1)^2$. Hence $d^2\mid 208$ or $d^2\mid 13\cdot 4^2$, so $d\in\{1,2,4\}$. Take $t=xy$ with $t\in\mathbb{Z}_+$.
Case I. If $d=1$, then $(xy)^2+208=4(xy+1)^2$ or $3t^2+8t-204=0$, without solutions.
Case II. If $d=2$, then $16x^2y^2+208=16(xy+1)^2$ or $t^2+13=t^2+2t+1\Rightarrow t=6$, so $(x,y)\in\{(1,6);(2,3);(3,2);(6,1)\}\Rightarrow(a,b)\in\{(2,12);(4,6);(6,4);(12;2)\}$.
Case III. If $d=4$, then $16^2x^2y^2+208=4\cdot 16(xy+1)^2$ or $16t^2+13=4(t+1)^2$ and if $t\in\mathbb{Z}$, then 13 must be even, contradiction!
Finally, the solutions are $(a,b)\in\{(2,12);(4,6);(6,4);(12;2)\}$.

A6 Let $x_i>1$, for all $i\in\{1,2,3,...,2011\}$. Prove the inequality $\sum_{i=1}^{2011}\frac{x_i^2}{x_{i+1}-1}\ge 8044$ where $x_{2012}=x_1$. When does equality hold?

Solution 1

Realize that $(x_i-2)^2\ge 0\Leftrightarrow x_i^2\ge 4(x_i-1)$. So we get:
$\frac{x_1^2}{x_2-1}+\frac{x_2^2}{x_3-1}+...+\frac{x_{2011}^2}{x_1-1}\ge 4\left(\frac{x_1-1}{x_2-1}+\frac{x_2-1}{x_3-1}+...+\frac{x_{2011}-1}{x_1-1}\right)$. By $AM-GM$:

$\dfrac{x_1-1}{x_2-1}+\dfrac{x_2-1}{x_3-1}+...+\dfrac{x_{2011}-1}{x_1-1} \geq 2011 \cdot \sqrt[2011]{\dfrac{x_1-1}{x_2-1}\cdot\dfrac{x_2-1}{x_3-1}\cdot...\cdot\dfrac{x_{2011}-1}{x_1-1}} = 2011.$

Finally, we obtain that $\dfrac{x_1^2}{x_2-1}+\dfrac{x_2^2}{x_3-1}+...+\dfrac{x_{2011}^2}{x_1-1} \geq 8044.$

Equality holds when $(x_i-2)^2=0,\ (\forall)\ i=\overline{1,2011}$, or $x_1=x_2=...=x_{2011}=2$.

Solution 2

All the denominators are greater than 0, so by $Cauchy-Schwarz$ we have:

$\dfrac{x_1^2}{x_2-1}+\dfrac{x_2^2}{x_3-1}+...+\dfrac{x_{2011}^2}{x_1-1} \geq \dfrac{(x_1+x_2+...+x_{2011})^2}{x_1+x_2+...+x_{2011}-2011}$. It remains to prove that $\dfrac{(x_1+x_2+...+x_{2011})^2}{x_1+x_2+...+x_{2011}-2011} \geq 8044$ or $\left(\sum_{i=1}^{2011} x_i\right)^2 + 4\cdot 2011^2 \geq 4\cdot 2011 \cdot \sum_{i=1}^{2011} x_i$ which is obviously true by $AM-GM$ for $\left(\sum_{i=1}^{2011} x_i\right)^2$ and $4\cdot 2011^2$.

Equality holds when $x_1+x_2+...+x_{2011}=4022$ and $\dfrac{x_1}{x_2-1}=\dfrac{x_2}{x_3-1}=...=\dfrac{x_{2011}}{x_1-1}$ or $x_i^2-x_i=x_{i-1}x_{i+1}-x_{i-1},\ (\forall)\ i=\overline{1,2011} \Rightarrow \sum_{i=1}^{2011} x_i^2 = \sum_{i=1}^{2011} x_i x_{i+2}$ where $x_{2012}=x_1$ and $x_{2013}=x_2$. This means that $x_1=x_2=...=x_{2011}$.

So equality holds when $x_1=x_2=...=x_{2011}=2$ since $x_1+x_2+...+x_{2011}=4022$.

A7 Let a,b,c be positive real numbers with $abc=1$. Prove the inequality:

$$\frac{2a^2+\frac{1}{a}}{b+\frac{1}{a}+1}+\frac{2b^2+\frac{1}{b}}{c+\dfrac{1}{b}+1}+\frac{2c^2+\frac{1}{c}}{a+\frac{1}{c}+1} \geq 3$$

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Solution 1

By $AM-GM$ we have $2x^2+\dfrac{1}{x}=x^2+x^2+\dfrac{1}{x} \geq 3\sqrt[3]{\dfrac{x^4}{x}}=3x$ for all $x>0$, so we have:

$\sum_{cyc}\dfrac{2a^2+\frac{1}{a}}{b+\frac{1}{a}+1} \geq \sum_{cyc}\dfrac{3a}{1+b+bc} = 3\left(\sum_{cyc}\dfrac{a^2}{1+a+ab}\right) \geq \dfrac{3(a+b+c)^2}{3+a+b+c+ab+bc+ca}.$

By $AM-GM$ we have $ab+bc+ca \geq 3$ and $a+b+c \geq 3$. But $3(a^2+b^2+c^2) \geq (a+b+c)^2 \geq 3(a+b+c)$. So $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca \geq 3+a+b+c+ab+bc+ca$.

Hence $\sum_{cyc}\dfrac{2a^2+\frac{1}{a}}{b+\frac{1}{a}+1} \geq \dfrac{3(a+b+c)^2}{3+a+b+c+ab+bc+ca} \geq \dfrac{3(a+b+c)^2}{(a+b+c)^2}=3.$

Solution 2

Denote $a=\dfrac{y}{x},\ b=\dfrac{z}{y}$ and $c=\dfrac{x}{z}$. We have $\dfrac{2a^2+\frac{1}{a}}{b+\frac{1}{a}+1}=\dfrac{\frac{2y^2}{x^2}+\frac{x}{y}}{\frac{z}{y}+\frac{x}{y}+1}=\dfrac{2y^3+x^3}{x^2(x+y+z)}.$

Hence $\sum_{cyc} \frac{2a^2+\frac{1}{a}}{b+\frac{1}{a}+1} = \frac{1}{x+y+z} \cdot \sum_{cyc} \frac{2y^3+x^3}{x^2} = \frac{1}{x+y+z} \cdot \left(x+y+z+2\sum_{cyc} \frac{y^3}{x^2} \right)$.

By *Rearrangements Inequality* we get $\sum_{cyc} \frac{y^3}{x^2} \geq x+y+z$.

So $\sum_{cyc} \frac{2a^2+\frac{1}{a}}{b+\frac{1}{a}+1} \geq \frac{1}{x+y+z} \cdot (3x+3y+3z) = 3$.

A8 Decipher the equality $(\overline{LARN}-\overline{ACA}) : (\overline{CYP}+\overline{RUS}) = C^{Y^P} \cdot R^{U^S}$ where different symbols correspond to different digits and equal symbols correspond to equal digits. It is also supposed that all these digits are different from 0.

Solution

Denote $x = \overline{LARN} - \overline{ACA}$, $y = \overline{CYP} + \overline{RUS}$ and $z = C^{Y^P} \cdot R^{U^S}$. It is obvious that $1823 - 898 \leq x \leq 9187 - 121$, $135 + 246 \leq y \leq 975 + 864$, that is $925 \leq x \leq 9075$ and $381 \leq y \leq 1839$, whence it follows that $\frac{925}{1839} \leq \frac{x}{y} \leq \frac{9075}{381}$, or $0,502... \leq \frac{x}{y} \leq 23,81...$ Since $\frac{x}{y} = z$ is an integer, it follows that $1 \leq \frac{x}{y} \leq 23$, hence $1 \leq C^{Y^P} \cdot R^{U^S} \leq 23$. So both values C^{Y^P} and R^{U^S} are ≤ 23. From this and the fact that $2^{2^3} > 23$ it follows that at least one of the symbols in the expression C^{Y^P} and at least one of the symbols in the expression R^{U^S} correspond to the digit 1. This is impossible because of the assumption that all the symbols in the set $\{C, Y, P, R, U, S\}$ correspond to different digits.

A9 Let $x_1, x_2, ..., x_n$ be real numbers satisfying $\sum_{k=1}^{n-1} \min(x_k; x_{k+1}) = \min(x_1, x_n)$.

Prove that $\sum_{k=2}^{n-1} x_k \geq 0$.

Solution 1

Case I. If $\min(x_1, x_n) = x_1$, we know that $x_k \geq \min(x_k; x_{k+1})$ for all $k \in \{1, 2, 3, ..., n-1\}$.

So $x_1 + x_2 + ... + x_{n-1} \geq \sum_{k=1}^{n-1} \min(x_k; x_{k+1}) = \min(x_1, x_n) = x_1$, hence $\sum_{k=2}^{n-1} x_k \geq 0$.

Case II. If $\min(x_1, x_n) = x_n$, we know that $x_k \geq \min(x_{k-1}; x_k)$ for all $k \in \{2, 3, 4, ..., n\}$.

So $x_2 + x_3 + ... + x_n \geq \sum_{k=1}^{n-1} \min(x_k; x_{k+1}) = \min(x_1, x_n) = x_n$, hence $\sum_{k=2}^{n-1} x_k \geq 0$.

Solution 2

Since $\min(a, b) = \frac{1}{2}(a+b-|a-b|)$, after substitutions, we will have:

. $\sum_{k=1}^{n-1} \frac{1}{2}(x_k + x_{k+1} - |x_k - x_{k+1}|) = \frac{1}{2}(x_1 + x_n - |x_1 - x_n|) \Leftrightarrow ...$

$2(x_2 + x_3 + ... + x_{n-1}) + |x_1 - x_n| = |x_1 - x_2| + |x_2 - x_3| + ... + |x_{n-1} - x_n|$.

As $|x_1 - x_2| + |x_2 - x_3| + ... + |x_{n-1} - x_n| \geq |x_1 - x_2 + x_2 - x_3 + ... + x_{n-1} - x_n| = |x_1 - x_n|$, we obtain the desired result.