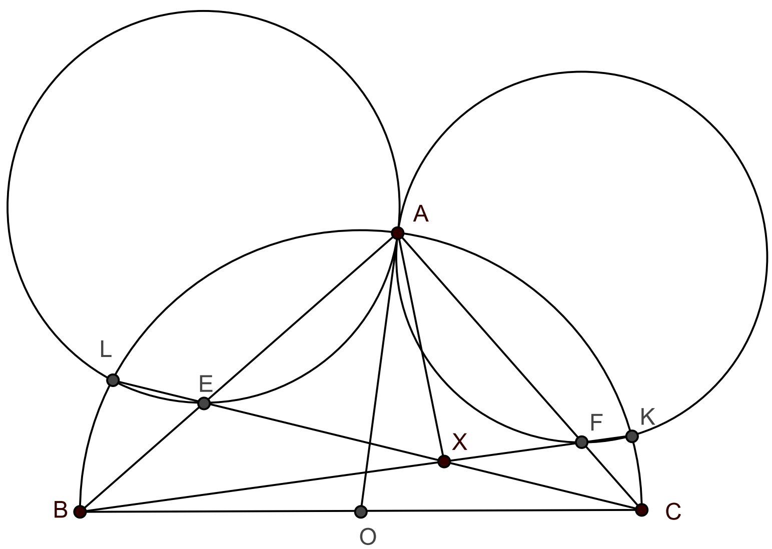
The fourth TST for JBMO 2013
Bucharest, May 24, 2013

Problem 1. Let A be a point on a semicircle of diameter $[BC]$, and X an arbitrary point inside the triangle ABC. The line BX intersects the semicircle for the second time in K, and intersects the line segment (AC) in F. The line CX intersects the semicircle for the second time in in L, and intersects the segment line (AB) in E. Prove that the circumcircles of triangles AKF and AEL are tangent.

Ioan-Laurenţiu Ploscaru

Solution: Denote by O the midpoint of the diameter $[BC]$. We prove that the line AO is tangent to both circumcircles. The conclusion follows readily.

As $AO = CO$ (radii), we have that $m(\angle CAO) = m(\angle ACO) = m(\angle AKB) = \frac{1}{2}m(\widehat{AF})$ (in the circumcircle of triangle AKF). It follows that AO is tangent to the circumcircle of triangle AKF. Similarly, AO is also tangent to the circumcircle of triangle AEL, hence the conclusion.



Problem 2. Let a, b, c be positive real numbers such that $a + b + c = 1$. Show that

$$\frac{1-a^2}{a+bc} + \frac{1-b^2}{b+ca} + \frac{1-c^2}{c+ab} \geq 6.$$

Lucian Petrescu

Solution 1: $\frac{1-a^2}{a+bc} + \frac{1-b^2}{b+ca} + \frac{1-c^2}{c+ab} \geq 6 \Leftrightarrow \left(\frac{1-a^2}{a+bc}+1\right) + \left(\frac{1-b^2}{b+ca}+1\right) + \left(\frac{1-c^2}{c+ab}+1\right) \geq 9 \Leftrightarrow$
$\frac{1-a^2+a+bc}{a+bc} + \frac{1-b^2+b+bc}{b+ca} + \frac{1-c^2+c+ab}{c+ab} \geq 9.$

From $a + b + c = 1$, we get $a - a^2 = ab + ac$ and two similar relations, hence it remains to be proven that

$$\frac{1+ab+bc+ca}{a+bc} + \frac{1+ab+bc+ca}{b+ca} + \frac{1+ab+bc+ca}{c+ab} \geq 9,$$

i.e.

$$[(a+bc)+(b+ca)+(c+ab)] \cdot \left(\frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}\right) \geq 9,$$

which follows immediately .

Equality holds when $a+bc=b+ca=c+ab$, which comes to $a=b=c=\frac{1}{3}$.

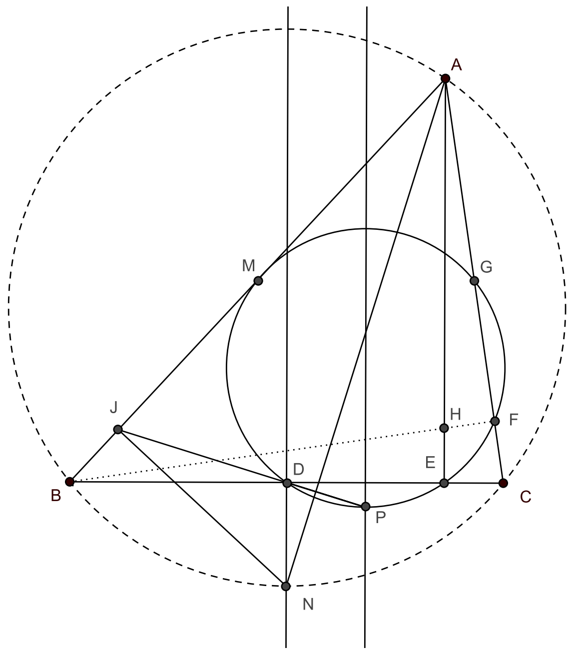
Solution 2: (Teodor Andrei Andronache) From $a+b+c=1$, we get $a+bc=a(a+b+c)+bc=(a+b)(a+c)$ and $1-a^2=(a+b+c)^2-a^2=(b+c)(2a+b+c)$. Denoting $a+b=x$, $b+c=y$, $c+a=z$, we have $\frac{1-a^2}{a+bc}=\frac{y(z+x)}{zx}=\frac{y}{x}+\frac{y}{z}$, and proceeding similarly for the two other fractions on the left hand side reduces the initial inequality to proving that $\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z} \geq 6$, which is clear.

Problem 3. Let D be the midpoint of the side $[BC]$ of the triangle ABC with $AB \neq AC$ and E the foot of the altitude from BC. If P is the intersection point of the perpendicular bisector of the segment line $[DE]$ with the perpendicular from D onto the the angle bisector of BAC, prove that P is on the Euler circle of triangle ABC.

Marius Bocanu

Solution: We may assume that $AB > AC$, the case when $AB < AC$ being similar.
Let $(AN$ be the angle bisector of angle BAC, and M the midpoint of side AB. As P is on the perpendicular bisector of segment DE, we have $DP = PE$, hence $m(\angle DPE) = 180° - 2m(\angle EDP) = 180° - 2m(\angle NAE) = 180° - 2\big(m(\angle NAC) - m(\angle EAC)\big) = 180° - m(\angle C) + m(\angle B)$.
On the other hand, $m(\angle DME) = m(\angle BME) - m(\angle BMD) = 180° - 2m(\angle B) - m(\angle A) = m(\angle C) - m(\angle B) = 180° - m(\angle DPE)$, which means that the quadrilateral $DPEM$ is cyclic. But points D, E, M lie on the Euler circle of triangle ABC, hence the conclusion.

Remark: If N is the intersection point between the angle bisector of angle A and the perpendicular bisector of segment BC, as shown in the figure below, and J is the intersection point of lines DP and AB, then it is easy to prove that the quadrilateral $BJDN$ is cyclic, hence NJ is perpendicular to AB. It follows that DP is in fact the Simson line corresponding to the point N. If H is the orthocenter of triangle ABC, then, according to problem 4 from the second test, the line DP bisects the segment line HN. It follows that P is the midpoint of the segment line HN and it is known that this midpoint belongs to the Euler circle of triangle ABC.



Problem 4. For any sequence $(a_1, a_2, ..., a_{2013})$ of integers, we call a triple (i, j, k) satisfying $1 \le i < j < k \le 2013$ to be *progressive* if $a_k - a_j = a_j - a_i = 1$. Determine the maximum number of *progressive* triples that a sequence of 2013 integers could have.

Ioan-Laurenţiu Ploscaru

Solution (by the author):
Consider the following operations one can apply to a sequence $(a_1, a_2, ..., a_{2013})$:
i) If $a_{n+1} < a_n,\ n = \overline{1, 2012}$ we swap the two numbers and we get $(a_1, a_2, ..., a_{n-1}, a_{n+1}, a_n, a_{n+2}, ..., a_{2013})$;
ii) If $a_{n+1} = a_n + d + 1$, where $n = \overline{1, 2012},\ d > 0$, then we add d to each of the numbers $a_1, a_2, ..., a_n$ and we obtain the sequence $(a_1 + d, a_2 + d, ..., a_n + d, a_{n+1}, ..., a_{2013})$.
Let m be the maximum number of *progressive* triples a sequence of 2013 integers can have. It is clear that applying the operation *i)* m cannot decrease, therefore we may assume that the sequence is increasing. Next, applying to this increasing sequence the operation *ii)*, again, m cannot decrease. It follows that it is enough to consider only sequences of the following type: $(\underbrace{a, ..., a}_{t_1}, \underbrace{a+1, ..., a+1}_{t_2}, ..., \underbrace{a+s-1, ..., a+s-1}_{t_s})$, where $t_1, t_2, ..., t_s$ are the number of occurrences of the different numbers, and $s \ge 3$.
But for such a sequence we have $m = t_1t_2t_3 + t_2t_3t_4 + ... + t_{s-2}t_{s-1}t_s$, where $s \ge 3$ and $t_1, t_2, \dots, t_s$ are positive integers having the sum 2013.
The maximum value of m is obtained for $s = 3$ or $s = 4$. Indeed, if $s \ge 5$, then replacing $t_1, t_2, ..., t_s$ with $t_2, t_3, (t_1 + t_4), ..., t_s$ we get a larger value of m.
For $s = 3$ we have $m = t_1t_2t_3 \le (t_1 + t_2 + t_3)^3/27 = 671^3$, therefore, in this case, the maximum value of m is 671^3 and it is obtained when $t_1 = t_2 = t_3 = 671$.
For $s = 4$ we have $m = (t_1 + t_4)t_2t_3 \le (t_1 + t_2 + t_3 + t_4)^3/27 = 671^3$, therefore, in this case, the maximum value of m is 671^3 and it is obtained when $t_1 + t_4 = t_2 = t_3 = 671$.
We conclude that $m = 671^3$.

Remark: (Vladimir Vîntu) In order to establish the maximum of $m = t_1t_2t_3 + t_2t_3t_4 + ... + t_{s-2}t_{s-1}t_s$, where $s \geq 3$ and $t_1, t_2, \ldots, t_s$ are non-negative integers having the sum 2013, one can notice that, according to the AM-GM inequality,

$$m \leq (t_1+t_4+\ldots)(t_2+t_5+\ldots)(t_3+t_6+\ldots) \leq \left(\frac{(t_1+t_4+\ldots)+(t_2+t_5+\ldots)+(t_3+t_6+\ldots)}{3}\right)^3$$

$$= 671^3.$$

As equality holds for example if $s = 3$ and $t_1 = t_2 = t_3 = 671$, the maximum value of m is indeed 671^3. One can readily find a complete description of the equality cases by examining when equality holds in both the estimations above.