The second TST for JBMO 2013
Bucharest, April 25, 2013

Problem 1. If $a, b, c > 0$ satisfy $a + b + c = 3$, then prove that

$$\frac{a^2(b+1)}{ab+a+b} + \frac{b^2(c+1)}{bc+b+c} + \frac{c^2(a+1)}{ca+c+a} \geq 2.$$

Mathematical Excalibur P322/Vol.14, no.2

Solution 1. We notice that

$$\frac{a^2(b+1)}{ab+a+b} = a - \frac{ab}{ab+a+b} \geq a - \frac{ab}{ab+2\sqrt{ab}} = a - 1 + \frac{2}{\sqrt{ab}+2}.$$

Writing two similar inequalities and adding them up, we obtain

$$\sum_{cyc} \frac{a^2(b+1)}{ab+a+b} \geq 2 \sum_{cyc} \frac{1}{\sqrt{ab}+2} \geq 2 \cdot \frac{(1+1+1)^2}{6+\sqrt{ab}+\sqrt{ac}+\sqrt{bc}} \geq \frac{18}{6+\frac{a+b}{2}+\frac{a+c}{2}+\frac{b+c}{2}} = 2.$$

Equality holds when $a = b = c = 1$.

Solution 2. We have, successively

$$\begin{aligned} \sum_{cyc} \frac{a^2(b+1)}{ab+a+b} &= \sum_{cyc} \frac{a^2}{\frac{ab+a+b}{b+1}} = \sum_{cyc} \frac{a^2}{a+1-\frac{1}{b+1}} \geq \frac{(a+b+c)^2}{a+b+c+3-\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right)} = \\ &= \frac{9}{6-\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right)} \geq \frac{9}{6-\frac{(1+1+1)^2}{a+1+b+1+c+1}} = \frac{9}{6-\frac{3}{2}} = 2. \end{aligned}$$

Equality holds when $a = b = c = 1$.

Solution 3: (Mathematical Excalibur) We notice that $\frac{a^2(b+1)}{ab+a+b} = a - \frac{ab}{ab+a+b}$. Applying the AM-GM inequality twice, we obtain $\frac{ab}{ab+a+b} \leq \frac{ab}{3\sqrt[3]{a^2b^2}} = \frac{\sqrt[3]{ab}}{3} \leq \frac{a+b+1}{9}$.

Writing the two similar inequalities and adding them up, yields the desired inequality. Equality holds when $a = b = c = 1$.

Problem 2. Call the number $\overline{a_1a_2\ldots a_m}$ ($a_1 \neq 0$, $a_m \neq 0$) the *reverese* of the number $\overline{a_m\ldots a_2a_1}$.
Prove that the sum between a number n and its *reverse* is a multiple of 81 if and only if the sum of the digits of n is a multiple of 81.

Andrei Eckstein

Solution: Consider $n = \overline{a_1a_2\ldots a_{m-1}a_m}$ and $r(n) = \overline{a_ma_{m-1}\ldots a_2a_1}$ its *reverse*. We have:

$$n + r(n) = \sum_{j=0}^{m}(a_j + a_{m-j}) \cdot 10^j = \sum_{j=0}^{m} a_j(10^j + 10^{m-j}).$$

Notice that $10^i + 10^{j+1} \equiv 10^j + 10^{i+1} \pmod{81}$, $\forall i, j \in \mathbb{N}$. It follows that there exists r such that $10^j + 10^{m-j} \equiv r$, $\forall j = \overline{0, m}$.
We obtain that $n + r(n) \equiv r \sum\limits_{j=0}^{m} a_j$. As $(r, 81) = 1$, the conclusion follows immediately.

Problem 3. The three-element subsets of a seven-element set are colored. If the intersection of two sets is empty then they have different colors. What is the minimum number of colors needed?

Solution 1: Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Two colors are not enough because the sets in the

following sequence of three-element subsets of A should have alternating colors: $\{1,2,3\}$, $\{4,5,6\}$, $\{7,1,2\}$, $\{3,4,5\}$, $\{6,7,1\}$, $\{2,3,4\}$, $\{5,6,7\}$, $\{1,2,3\}$.
With three colors we can color, for example, the three-element subsets of A as follows: we use
- the first color for all the subsets containing the element 7;
- a second color for all the subsets that do not contain the element 7 and for which the sum of their elements is even;
- a third color for all the remaining subsets.

Solution 2. *(Dan Schwarz)* We prove that the minimum number is 3. Assuming that a coloring with two colors was possible, let U be the set of the three-element subsets colored with the first color, and V be the set of the three-element subsets colored with the second color. For every subset $X \in U$ there are 4 subsets of $A \setminus X$ that have to be in V. If we represent the set of the three-element subsets as a graph in which we join two vertices (representing two three-element subsets of A) if they are disjoint, then every element of U is joined with exactly 4 elements of V, and vice-versa. It follows that the total number of edges, i.e. the numbers of pairs of disjoint three-element subsets, is $4 \operatorname{card} U = 4 \operatorname{card} V$, hence $\operatorname{card} U = \operatorname{card} V$. But then $\operatorname{card} U \cup V$ should be even, while in fact it is $\binom{7}{3} = 35$, which is odd, contradiction.

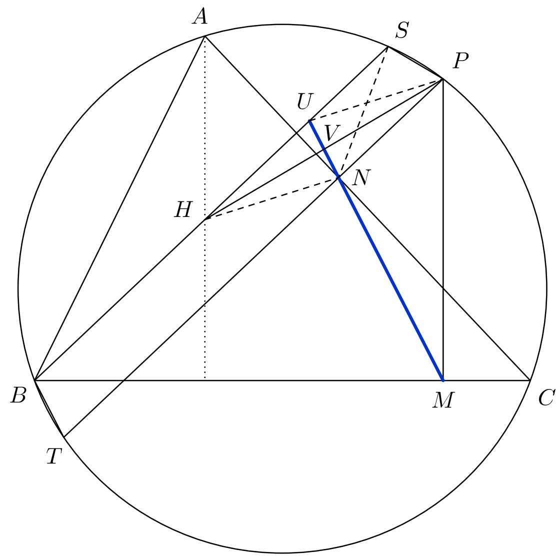
A coloring with three colors can be found as follows: we choose two arbitrary elements $x, y \in A$. We color the three-element subsets that do not contain neither x nor y with the first color, those that contain x but do not contain y with the second color, and finally the subsets that contain y with the third color.

Problem 4. Let H be the orthocenter of an acute-angled triangle ABC and P a point on the circumcenter of triangle ABC. Prove that the Simson line of P bisects the segment $[PH]$.

Solution: We consider P on the arc $\widehat{AC}$ not containing B. The other cases are similar. Let M, N be the projections of P onto BC, and CA, respectively, and T be the intersection point of $(PN$ with the circumcircle $\mathcal{C}$ of triangle ABC. Denote $\{S\} = BH \cap \mathcal{C}$, $\{U\} = MN \cap BH$ and V the midpoint of $[PH]$. S is the reflection of H across AC, hence $NH = NS$.

Notice that lines BT and MN are parallel. Indeed, quadrilaterals $CPNM$ and $CPBT$ being cyclic, it follows that $\angle BMN \equiv \angle CPN \equiv \angle CBT$. Then $BTNU$ is a parallelogram, while $UNPS$ is an isosceles trapezoid (or a rectangle).

It follows that $UP = NS = NH$. Then $HUPN$ is a parallelogram, and the midpoint V of the diagonal HP is also on the other diagonal, i.e. on the Simson line of P.


A
S
P
U
V
N
H
B
M
C
T