The fourth TST for JBMO 2013
Braşov, April 4, 2013

Problem 1. Let $a, b, c, d > 0$ satisfying $abcd = 1$. Prove that

$$\frac{1}{a+b+2} + \frac{1}{b+c+2} + \frac{1}{c+d+2} + \frac{1}{d+a+2} \leq 1.$$

Gheorghe Eckstein

Solution:
We have $\dfrac{1}{a+b+2} + \dfrac{1}{c+d+2} \leq \dfrac{1}{2\sqrt{ab}+2} + \dfrac{1}{2\sqrt{cd}+2}$. Denoting $\sqrt{ab} = x$, we get $\sqrt{cd} = \dfrac{1}{x}$, and the sum on the right hand side in the inequality above is $\dfrac{1}{2}\left(\dfrac{1}{x+1} + \dfrac{x}{1+x}\right) = \dfrac{1}{2}$.

Proceeding similarly with the two other terms of the sum, we obtain the desired inequality. Equality holds when $a = b = c = d = 1$.

Problem 2. Weights of 1 g, 2 g, ..., 200 g are placed on the two pans of a balance such that on each pan there are 100 weights and the balance is in equilibrium. Prove that one can swap 50 weights from one pan with 50 weights from the other pan such that the balance remains in equilibrium.

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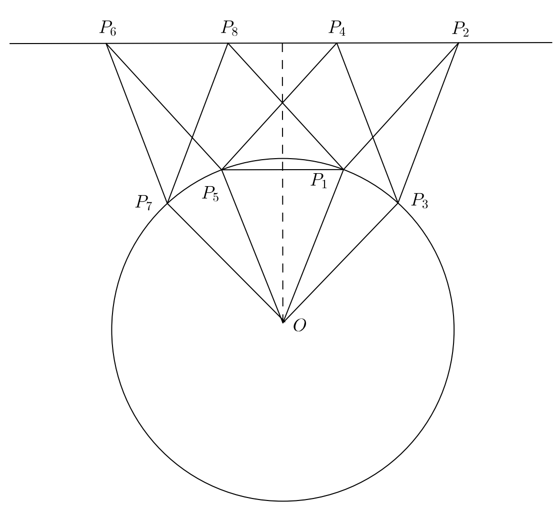
Solution:
We call a pair two weights whose sum is 201g. We wish to obtain, in the end, 50 pairs on each of the two pans of the balance.
If on the pan on the left we have the weights a_1, a_2, ..., a_{50} and their pairs b_1, b_2, ..., b_{50} are on the pan on the right, we move the weights such that, in the end, on the left pan we have the weights a_1, a_2, ..., a_{50} together with their pairs, b_1, b_2, ..., b_{50}.
If we have less then 50 pairs that are split between the two pans, we must have at least 25 pairs on the left pan and (at least) 25 pairs on the right pan. Moving 25 complete pairs from the right pan next to 25 pairs from the left pan, we obtain, again, 50 complete pairs on one pan, hence the desired result.

Problem 3. Consider a circle centered at O with radius r and a line ℓ not passing through O. A grasshopper is jumping to and fro between the points of the circle and the line, the length of each jump being r. Prove that there are at most 8 points for the grasshopper to reach.
Solution.

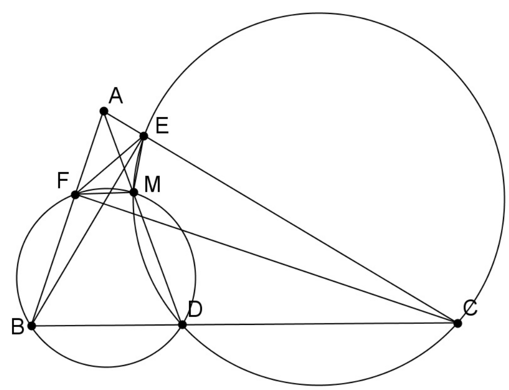


We assume that, when having the choice between only two places to jump to, the grasshopper never jump back to the point from which he got to that place. Let us denote by P_1 the starting point of the grasshopper, with P_2 the point on the line on which he has jumped from P_1, and so on. As the length of the jumps are all equal to r, $OP_1P_2P_3$ is a rhombus (possibly a degenerate one). Similarly, $OP_3P_4P_5$ is also a rhombus. It follows that the triangles P_1OP_5 and $P_2P_3P_4$ are congruent (SAS), and from here we obtain that P_1P_5 is parallel to ℓ. We deduce that P_5 is the reflection of P_1 across the perpendicular line from O onto ℓ. (This fact remains true even in the degenerate cases.) From P_5, the grasshopper can get to P_9 which, as above, is the reflection of P_5 across the perpendicular line from O onto ℓ, i.e. P_1. In conclusion, the grasshopper can reach only the points P_k, $k = \overline{1, 8}$ (which are not necessarily distinct).

Problem 4. In the acute-angled triangle ABC, with $AB \neq AC$, D is the foot of the angle bisector of angle A, and E, F are the feet of the altitudes from B and C, respectively. The circumcircles of triangles DBF and DCE intersect for the second time at M. Prove that $ME = MF$.

Leonard Giugiuc

Solution.



Triangles AEF and ABC are similar, therefore $AF \cdot AB = AE \cdot AC$. It follows the point A is on the radical axis of the two circumcircles, hence $M \in AD$. We have that $m(\angle EMF) = 360^\circ - (180^\circ - m(\angle FBD)) - (180^\circ - m(\angle ECD)) = m(\angle B) + m(\angle C) = 180^\circ - m(\angle A)$; it follows that the quadrilateral $AEMF$ is cyclic. This means that $\angle MEF \equiv \angle FAM$ and $\angle MFE \equiv \angle EAM$, i.e. triangle MEF is isosceles, with $ME = MF$.

Problem 5. a) Prove that for every positive integer n, there exist $a, b \in \mathbb{R} \setminus \mathbb{Z}$ such that the set

$$A_n = \{a - b,\ a^2 - b^2,\ a^3 - b^3,\ \ldots,\ a^n - b^n\}$$

contains only positive integers.
b) Let a and b be two real numbers such that the set

$$A = \{a^k - b^k |\ k \in \mathbb{N}^*\}$$

contains only positive integers. Prove that a and b are integers.

Solution.

a) For $a = 2^{n-1} + \dfrac{1}{2}$ and $b = \dfrac{1}{2}$ we have

$a^k - b^k = 2^{n-1} \cdot \dfrac{(2^n+1)^{k-1} + (2^n+1)^{k-2} + \cdots + (2^n+1) + 1}{2^{k-1}} =$
$2^{n-k} \cdot \left((2^n+1)^{k-1} + (2^n+1)^{k-2} + \cdots + (2^n+1) + 1\right) \in \mathbb{N}^*$ for all $k \in \{1, 2, \ldots, n\}$.

b) If $a - b = k_1 \in \mathbb{N}^*$ and $(a-b)(a+b) = k_2 \in \mathbb{N}^*$, then $a = \dfrac{k_2 + k_1^2}{2k_1}$ and $b = \dfrac{k_2 - k_1^2}{2k_1}$.
The greatest common divisor of $k_2 + k_1^2$ and $2k_1$ is the same as that of $k_2 - k_1^2$ and $2k_1$, hence a and b are rational numbers which, in their reduced form, have the same denominator.
Put $a = \dfrac{p}{q}$ and $b = \dfrac{r}{q}$, where $(p,q) = 1$ and $(r,q) = 1$. We have:

$a^n - b^n = \dfrac{(p-r)\left(p^{n-1} + p^{n-2}r + \cdots + pr^{n-2} + r^{n-1}\right)}{q^n} =$
$\dfrac{(p-r)\left(p^{n-1} - r^{n-1} + \left(p^{n-2} - r^{n-2}\right)r + \left(p^{n-3} - r^{n-3}\right)r^2 + \cdots + (p-r)\,r^{n-2} + nr^{n-1}\right)}{q^n} =$
$\dfrac{(p-r)\left(M_{q^{n-1}} + M_{q^{n-2}} + \cdots + M_{q^2} + M_q + nr^{n-1}\right)}{q^n} \in \mathbb{N}^*.$

There exist n_0, m, s with $(m,q) = 1$, such that, for all $n \geq n_0$, we have
$a^n - b^n = \dfrac{m\left(M_{q^{n-1}} + M_{q^{n-2}} + \cdots + M_{q^2} + M_q + nr^{n-1}\right)}{sq^{n-n_0}}.$
It follows that $q \mid n$ for all $n > n_0$, hence $q = 1$, and $a, b \in \mathbb{Z}$.