First JTST 2008

1. Find all integers (x, y, z) satisfying the equality:

$$x^2(y-z)+y^2(z-x)+z^2(x-y)=2.$$

2. Positive real numbers a, b, c satisfy the inequality $a+b+c \leq \frac{3}{2}$. Find the smallest possible value of:

$$S=abc+\frac{1}{abc}.$$

3. Rhombuses $ABCD$ and $A_1B_1C_1D_1$ are equal. Side BC intersects sides B_1C_1 and C_1D_1 at points M and N respectively. Side AD intersects sides A_1B_1 and A_1D_1 at points Q and P respectively. Let O be the intersection point of lines MP and QN. Find $m(\sphericalangle A_1B_1C_1)$, if $m(\sphericalangle QOP)=\frac{1}{2}m(\sphericalangle B_1C_1D_1)$.

4. The square table 10×10 is divided in squares 1×1. In each square 1×1 is written one of the numbers $\{1, 2, 3, \ldots, 9, 10\}$. Numbers from any two adjacent or diagonally adjacent squares are coprime. Prove, that there exists a number, which is written in this table at least 17 times.

Second JTST 2008

5. Find all natural pairs (x, y), such that x and y are coprime and satisfy the equality: $2x^2+5xy+3y^2=41x+62y+21$.

6. Solve the equation $2(x^2-3x+2)=3\sqrt[3]{x^3+8}$, where $x \in \mathbb{R}$.

7. In an acute triangle ABC, points A_1, B_1, C_1 are the midpoints of the sides BC, AC, AB, respectively. It is known that $AA_1=d(A_1, AB)+d(A_1, AC)$, $BB_1=d(B_1, AB)+d(A_1, BC)$, $CC_1=d(C_1, AC)+d(C_1, BC)$, where $d(X, YZ)$ denotes the distance from point X to the line YZ. Prove, that triangle ABC is equilateral.

8. An archipelago consists of n islands: $I_1, I_2, \ldots, I_n$ and $a_1, a_2, \ldots, a_n$ - number of the roads on each island. $a_1=55$, $a_k=a_{k-1}+(k-1)$, $(k=2, 3, \ldots, n)$.
a) Does there exist an island with 2008 roads?
b) Calculate $a_1+a_2+\ldots+a_n$.

Third JTST 2008

9. Find all triplets (x, y, z), that satisfy:

$$\begin{cases} x^2 - 2x - 4z = 3 \\ y^2 - 2y - 2x = -14 \\ z^2 - 4z - 4y = -18. \end{cases}$$

10. Solve in prime numbers:

$$\begin{cases} 2a - b + 7c = 1826 \\ 3a + 5b + 7c = 2007. \end{cases}$$

11. Let $ABCD$ be a convex quadrilateral with $AD = BC$, $CD \nparallel AB$, $AD \nparallel BC$. Points M and N are the midpoints of the sides CD and AB, respectively.
a) If E and F are points, such that $MCBF$ and $ADME$ are parallelograms, prove that $\Delta\, BFN \equiv \Delta\, AEN$.
b) Let $P = MN \cap BC$, $Q = AD \cap MN$, $R = AD \cap BC$. Prove that the triangle PQR is iscosceles.

12. A natural nonzero number, which consists of m digits, is called *hiperprime*, if its any segment, which consists of $1, 2, \ldots, m$ digits is prime (for example 53 is hiperprime, because numbers 53, 3, 5 are prime). Find all hiperprime numbers.